Problem Set 2 – due September 17

See the course website for policy on collaboration.

Notation Throughout this problem set, k denotes an algebraically closed field. For R a finitely generated k algebra, we write $\operatorname{MaxSpec} R$ for the set of k-algebra maps $R \to k$. Since we have now proved the Nullstellansatz, you now know that this is also the set of maximal ideals of R (hence Max). For $\phi : R \to S$ a map of k-algebras, we write ϕ^* for the induced map $\operatorname{MaxSpec} S \to \operatorname{MaxSpec} R$.

Problem 1 Let A be an $n \times n$ matrix with entries in k. Let R be the commutative ring $k[A]$. Explain a relationship between the eigenvalues of A and $\operatorname{MaxSpec} R$. The set of eigenvalues of A is often called the ***spectrum*** of A, explaining the Spec part of the name.

Remark: Those of you who know quantum mechanics will know that the spectral lines of a hot gas are determined by the eigenvalues of the Schrödinger operator. This is actually a fortuitous coincidence! The terminology "spectrum" for the eigenvalues of a matrix was introduced by Wertinger "Beiträge zu Riemanns Integrationsmethode für hyperbolische Differentialgleichungen, und deren Anwendungen auf Schwingungsprobleme" (1897), thirty years before Schrödinger!

Problem 2 Let R be a finitely generated k-algebra. Choose generators $x_1, x_2, \ldots, x_n$ and write $R = k[x_1, \ldots, x_n]/I$. So we have a natural bijection $\operatorname{MaxSpec}(R) \longleftrightarrow Z(I)$.

Show that $Y \subseteq X$ is closed in the Zariski topology if and only if there is an ideal $J \subseteq R$ such that Y corresponds to the set of maximal ideals containing J. Thus, we can describe the topology induced on $\operatorname{MaxSpec} R$ by the Zariski topology without reference to the choice of generators of R.

Problem 3 Let $X \subseteq k^m$ be Zariski closed, and let $f : X \to k^n$ be a regular map. Define the graph of f, written $\Gamma(f)$, to be the set $\{(x, y) \in X \times k^n : f(x) = y\}$.

(a) Show that $\Gamma(f)$ is Zariski closed in $k^m \times k^n$.

(b) Show that $\Gamma(f) \cong X$. That is to say, show that there are regular maps $\Gamma(f) \to X$ and $X \to \Gamma(f)$ which are mutually inverse.

Problem 4 Describe the images of the following maps. Are they open? Closed?

(a) Map $\{(x, y) \in k^2 : xy = 1\}$ to k by $(x, y) \mapsto x$.

(b) Map k^2 to k^2 by $(x, y) \mapsto (x, xy)$.

(c) Let $SL_2 = \{\left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right) : \ wz - xy = 1\}$. Map SL_2 to k^2 by $\left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right) \mapsto (w, x)$.

Problem 5 On the previous problem set, we showed that the map $k \to \{(x, y) : y^2 = x^3\}$ given by $t \mapsto (t^2, t^3)$ is bijective, but its inverse is not regular. Give an example of a Zariski closed subset X of k^2, and a regular bijection $X \to k$, whose inverse is not regular.

Problem 6 Let D be an integral domain and K its field of fractions. Suppose that g_1 and g_2 are nonzero elements of D and that the ideal $\langle g_1, g_2 \rangle$ is all of D. Let x be an element of K which can be represented as f_1/g_1 and as f_2/g_2, for some f_1 and $f_2 \in D$. Show that $x \in D$.

Problem 7 This problem asks you to play a bit more with resultants, a technical trick which we introduced in order to prove the Nullstellansatz. Recall that, given two polynomials $f := f_d z^d + f_{d-1} z^{d-1} + \cdots + f_1 z + f_0$ and $g := g_e z^e + g_{e-1} z^{e-1} + \cdots + g_1 z + g_0$ over k, the resultant

$R(f, g)$ is defined to be

$$\det \begin{pmatrix} f_d & f_{d-1} & \cdots & f_2 & f_1 & f_0 & & & \\ & f_d & f_{d-1} & \cdots & f_2 & f_1 & f_0 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & f_d & f_{d-1} & \cdots & f_2 & f_1 & f_0 \\ g_e & g_{e-1} & \cdots & g_1 & g_0 & & & & \\ & g_e & g_{e-1} & \cdots & g_1 & g_0 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & g_e & g_{e-1} & \cdots & g_1 & g_0 & \\ & & & & g_e & g_{e-1} & \cdots & g_1 & g_0 \end{pmatrix}$$

where the first parallelogram has e rows and the second parallelogram has d.

(a) Show that, if f_d and g_e are both nonzero, then $R(f, g) = 0$ if and only if f and g have a common root.

(b) More generally, show that $R(f, g) = 0$ if and only if either (i) f and g have a common root or (ii) $f_d = g_e = 0$.

The above problem might make you wonder whether there is some better function $S(f, g)$ which would vanish when f and g have a common root, but not when $f_d = g_e = 0$. the point of part (c) is to show that the answer is "no". More precisely:

(c) Let S be a polynomial in the variables f_d, f_{d-1}, $\ldots$, f_1, f_0, g_e, g_{e-1}, $\ldots$, g_1, g_0 and that $S(f_d, \ldots, f_0, g_e, \ldots, g_0)$ vanishes whenever $f_d z^d + f_{d-1} z^{d-1} + \cdots + f_1 z + f_0$ and $g_e z^e + g_{e-1} z^{e-1} + \cdots + g_1 z + g_0$ have a common root. Show that S must also vanish whenever $f_d = g_e = 0$.

Problem 8 We work inside the ring $k[u, v, w, x, y, z]$. We write M for the matrix

$$M := \begin{pmatrix} 0 & u & v & w \\ -u & 0 & x & y \\ -v & -x & 0 & z \\ -w & -y & -z & 0 \end{pmatrix}.$$

Let I_4 be the ideal generated by $\det M$ and let I_3 be the ideal generated by the determinants of the sixteen 3×3 submatrices of M.

(a) One of the containments $I_3 \subset I_4$, $I_4 \subset I_3$ is true and the other is false; which is which?

(b) Letting $I_a \subsetneq I_b$ denote the true relation, show that $I_a \subsetneq I_b \subsetneq \sqrt{I_a}$.

(c) Show that $\sqrt{I_a} = \sqrt{I_b}$.

(d) What fun fact about 4×4 skew symmetric matrices with entries in k have you proved?