Problem Set 3 – due September 24

See the course website for policy on collaboration.

Notation The symbol k denotes an algebraically closed field. Let X be a Zariski closed subset of k^m, let $\Omega \subseteq X$ be Zariski open and let f be a function $\Omega \to k$. For $x \in \Omega$ we say that f is ***regular at*** x if there is a Zariski open neighborhood U of x and regular functions g and h on X such that $h|_U \neq 0$ and $f|_U = (g/h)|_U$. We say that f is regular on Ω if it is regular at every point of Ω. We write $\mathbb{A}^n$ for k^n when we want to think of it as a geometric space. An ***affine variety*** X is a Zariski closed subset of some $\mathbb{A}^n$; a ***quasi-affine variety*** is a Zariski open subset of some affine variety. For us, the word "variety" does not imply irreducible.

Problem 1 Let X be an affine variety. Show that the Zariski topology on X is compact[1].

Warning: Your intuitions about the consequences of this result are probably false: If $k = \mathbb{C}$, this doesn't imply that regular functions $X \to \mathbb{C}$ are bounded, because they are not continuous using the Zariski topology on X and the analytic topology on $\mathbb{C}$. If $f : X \to Y$ is regular, this doesn't imply $f(X)$ is closed, because Y isn't Hausdorff.

Problem 2 Let k have characteristic not equal to 2. Let

$$O_2 = \left\{ \begin{pmatrix} w & x \\ y & z \end{pmatrix} : \begin{pmatrix} w & x \\ y & z \end{pmatrix} \begin{pmatrix} w & y \\ x & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

(a) Show that O_2 is disconnected, explicitly writing it disjoint union of two closed components.

(b) Find an idempotent in $\mathcal{O}_{O_2}$ other than 0 and 1.

Problem 3 Let X be $\{(w, x, y, z) : \ wy = x^2, \ xz = y^2\}$.

(a) Show that X is not irreducible, explicitly writing $X = X_1 \cup X_2$ for two Zariski closed sets X_1 and X_2 with $X_1, X_2 \neq X$.

(b) Show that the X_1 and X_2 you found are irreducible. (Hint: A subring of a domain is a domain. Alternate hint: The image of an irreducible variety under a regular map is irreducible.)

(c) What is the relation between the two hints? *I.e.* if A is a subring of B, is there a surjection $\operatorname{MaxSpec} B \to \operatorname{MaxSpec} A$? What about *vice versa*?

Problem 4 Repeat problem 3, parts (a) and (b) for

$$X = \left\{ \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix} : \det \begin{pmatrix} u & v \\ x & y \end{pmatrix} = \begin{pmatrix} v & w \\ y & z \end{pmatrix} = 0 \right\}$$

Problem 5 Let X be an affine variety. Show that the units in the ring $\mathcal{O}_X$ are precisely the regular functions $f : X \to k$ which are nowhere 0.

Problem 6 Let $X = \mathbb{A}^2$, with coordinate functions x and y. Let $\Omega = X \setminus \{(0, 0)\}$. Show that any regular function on Ω extends to a regular function on X.

Problem 7 Let $X = \{(w, x, y, z) : wy = wz = xy = xz = 0\}$. Let $\Omega = X \setminus \{(0, 0, 0, 0)\}$.

(a) Decompose X into irreducible components. You should find two, call them X_0 and X_1.

(b) Show that $X_0 \cap \Omega$ and $X_1 \cap \Omega$ are disjoint.

(c) Let $f : \Omega \to k$ be 0 on X_0 and 1 on X_1. Show that f is regular on Ω.

(d) Show that any function $h \in \mathcal{O}_X$ which is nonzero on Ω is constant. (I.e. equal to some $c \in k$.) Show that f cannot be written in the form g/h with g and $h \in \mathcal{O}_X$ and $h|_\Omega \neq 0$.

---

[1] Compact here means that any open cover has a finite subcover; it does not mean that X is Hausdorff. Bourbaki and some other French textbooks call the open cover condition "quasi-compact" and use "compact" to mean "quasi-compact and Hausdorff". In my experience, if you say "compact" in the English speaking mathematical word, people will assume that you mean just the open cover definition; if you say "quasi-compact" people will understand you to mean the same definition, but will be warned that you are particularly focusing on non-Hausdorff examples.