Problem Set 4 – due October 1

See the course website for policy on collaboration.

Definitions/Notation For a vector space V, we write $V_{\neq 0}$ for $V \setminus \{0\}$ and we write $\mathbb{P}(V) = V_{\neq 0}/k^{\times}$. If $(x_0, x_1, \ldots, x_n)$ are coordinates on V, we write $(x_0 : x_1 : \cdots : x_n)$ for the homogenous coordinates on $\mathbb{P}(V)$. For $X \subset \mathbb{P}(V)$, we write $CX_{\neq 0}$ for the points in $V_{\neq 0}$ lying above $CX_{\neq 0}$ and we write CX for $CX_{\neq 0} \cup \{0\}$. If L is a codimension one linear space in V *not* passing through 0, then the composite $L \hookrightarrow V_{\neq 0} \twoheadrightarrow \mathbb{P}(V)$ is injective; the image of such a map is a ***linear chart***.

The ***Zariski topology*** on $\mathbb{P}(V)$ is the quotient of the Zariski topology on $V_{\neq 0}$, meaning that $Z \subseteq \mathbb{P}(V)$ is ***Zariski closed*** if and only if $CZ_{\neq 0}$ is Zariski closed in $V_{\neq 0}$. This is equivalent to:

1. Z can be defined by the vanishing of a set of homogenous polynomials.
2. CZ is Zariski closed in the affine space V.
3. For every linear chart L, $Z \cap L$ is Zariski closed in the affine space L. It is enough to check this for enough linear charts to cover $\mathbb{P}(V)$.

A Zariski closed subset of $\mathbb{P}^n$ is a ***projective variety***. An open subset of a projective variety is a ***quasi-projective variety***. Regular functions on quasi-projective varieties are defined by Problem 3. You may the results of Problem 3 in the following problems. A ***regular map*** of quasi-projective varieties is defined by Problem 5. You may use the result of Problem 5 in Problem 6.

Problem 1 (This problem could have been on Problem Set 1, but I didn't get to it until now.) Let X and Y be Zariski closed in $\mathbb{A}^m$ and $\mathbb{A}^n$ respectively. We'll write π_X and π_Y for the projections $X \times Y \to X$ and $X \times Y \to Y$ respectively.

(a) Show that $X \times Y$ is Zariski closed in $\mathbb{A}^m \times \mathbb{A}^n$.
(b) Show that $\mathcal{O}_{X \times Y}$ is generated by the pullbacks of $\mathcal{O}_X$ and $\mathcal{O}_Y$ along π_X and π_Y respectively.

Problem 2 Two conics should intersect at 4 points. So, what are the 4 points of $\mathbb{P}^2$ where the circles $(x-3)^2 + y^2 = 25$ and $(x+3)^2 + y^2 = 25$ meet?

Problem 3 Let X be closed in $\mathbb{P}^n$ and Ω be open in X. We consider three conditions:

1. For every $x \in \Omega$, there is an open set U with $x \in U \in \Omega$ and homogenous polynomials g and h of the same degree such that $h|_U \neq 0$ and $(g/h)|_U = f|_U$.
2. The pullback of f to the quasi-affine $C\Omega_0$ is regular.
3. For every linear chart L, the restriction $f|_{L \cap \Omega}$ is regular on the quasi-affine $L \cap \Omega$.

(a) Show that (1) implies (2).
(b) Show that (2) implies (3).
(c) Show that (3) implies (1).
We define f to be ***regular*** if these conditions hold.

Problem 4 Prove that the only regular functions $\mathbb{P}^1 \to k$ are the constants.

Problem 5 Let X and Y be quasi-projective varieties and let $\phi : X \to Y$ be a continuous function. Suppose we have open covers U_i and V_i of X and Y by affine varieties such that $\phi(U_i) \subseteq V_i$ and $\phi : U_i \to V_i$ is regular. (Since these are affine varieties, we know what this means.)

Let U and V be any other open affine varieties in X and Y, with $\phi(U) \subseteq V$. Show that $\phi : U \to V$ is regular. (There is very little to do here; this is a matter of chasing symbols.)

In this case, we say that ϕ is ***a regular map***.

Problem 6 Map $\mathbb{P}^2$ to $\mathbb{P}^5$ by $\phi : (p : q : r) \mapsto (p^2 : pq : pr : q^2 : qr : r^2)$. We write $(u : v : w : x : y : z)$ for the coordinates on $\mathbb{P}^5$.

(a) Show that ϕ is injective.
(b) Show that the image of ϕ is closed, and give explicit homogenous equations for the image.
(c) Show that the inverse map $\phi(\mathbb{P}^2) \to \mathbb{P}^2$ is regular.