Problem Set 5 – due ~~October 8~~ October 10

See the course website for policy on collaboration.

Definitions/Notation As a set, the product of two quasi-projective varieties X and Y, is the product of the sets X and Y.

If X and Y are affine, the ring of regular functions on $X \times Y$ is the ring generated by the pullbacks of regular functions from the two factors; $Z \subset X \times Y$ is closed if Z is of the form $f_1 = f_2 = \cdots = f_s = 0$ for regular functions $f_1, f_2, \ldots, f_s$ on $X \times Y$.

If X and Y are quasi-projective, then $Z \subset X \times Y$ is ***closed*** if, for some (equivalently any) open covers U_i and V_i of X and Y by affines, the sets $Z \cap (U_i \times V_i)$ are closed in $U_i \times V_i$. Given $\Omega \subset Z$ and $f : \Omega \to k$, we say that f is ***regular*** on Ω if, for some (equivalently any) open covers U_i and V_i of X and Y by affines, the function $f|_{\Omega \cap (U_i \times V_i)}$ is regular on $\Omega \cap (U_i \times V_i)$.

Alternatively, we may define the topology and regular functions on $X \times Y$ using the Segre embedding $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{mn-1}$. You may use either definition on this problem set.

Problem 1 (This problem could have appeared on the second problem set, but I need it now.) Let A be a commutative ring and I and J ideals. Then $[I : J]$ is defined by

$$[I : J] := \{f \in A : fj \in I \text{ for all } j \in J\}.$$

The ideal $[I : J^\infty]$, called the ***saturation of I with respect to J*** is defined to be

$$[I : J^\infty] = \bigcup_{n=0}^{\infty} [I : J^n].$$

(a) Let I and $J \subset k[x_1, \ldots, x_n]$ be radical ideals, with $I = I(X)$ and $J = I(Y)$. Show that $[I : J]$ is radical, $[I : J] = [I : J^\infty]$ and $[I : J] = I(\overline{X \setminus (X \cap Y)})$. Here $\overline{S}$ means the Zariski closure of S.

(b) Let I and $J \subset k[x_1, \ldots, x_n]$ be ideals, not necessarily radical, with $X = Z(I)$ and $Y = Z(J)$. Show that $Z([I : J^\infty]) = \overline{X \setminus (X \cap Y)}$. Show that we need not have $Z([I : J]) = \overline{X \setminus (X \cap Y)}$. (Extremely explicit hint: Take $I = \langle x^2 \rangle$ and $J = \langle x \rangle$.)

Problem 2 Let $U \subset \mathbb{P}^2$ be the complement of the conic $p^2 + q^2 + r^2 = 0$. In this problem, we will show that U is isomorphic to an affine variety.

Embed $\mathbb{P}^2$ into $\mathbb{P}^5$ by $\phi : (p : q : r) \mapsto (p^2 : pq : pr : q^2 : qr : r^2)$. In the previous problem set, you found that $\phi(\mathbb{P}^2)$ is a closed subset of $\mathbb{P}^5$, with equations $ux = v^2$, $uy = w^2$, $xz = y^2$, $uy = vw$, $vz = wy$ and $wx = vy$.

(a) Show that $\phi(U)$ lies in a linear chart of $\mathbb{P}^5$, and is closed in that linear chart.

(b) Give explicit generators and relations for the ring of regular functions on U.

In general, this method shows that $\mathbb{P}^N \setminus \{F = 0\}$ is affine for any homogenous polynomial F.

Problem 3 Identify $\mathbb{A}^{10}$ with the set of homogenous cubic polynomials in three variables, identifying the point $(a_{300}, a_{210}, a_{201}, \cdots, a_{003})$ with $a_{300}x^3 + a_{210}x^2y + a_{201}x^2z + \cdots + a_{003}z^3$. Show that the set of polynomials which factor as (linear)(quadratic) is Zariski closed in $\mathbb{A}^{10}$. (To be clear, this includes the cases where the quadratic factors further as (linear)(linear), and the 0 polynomial. You might find it easier to first think in $\mathbb{P}^9$ rather than $\mathbb{A}^{10}$.)

Problem 4 Let X be a quasi-projective variety:

(a) Show that the diagonal $\{(x, x)\}$ is closed in $X \times X$.

(b) Let W be a quasi-projective variety and let ϕ_1 and ϕ_2 be two regular maps $W \to X$. Suppose that U is a dense subset of W and $\phi_1|_U = \phi_2|_U$. Show that $\phi_1 = \phi_2$. (Hint: This is pure topology.)

Problem 5 The ***blow up of*** $\mathbb{A}^n$, denoted $B\ell_0\mathbb{A}^n$, is defined to be the subset of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ given by the equations $x_iy_j - x_jy_i = 0$, where $(x_1, \ldots, x_n)$ are the coordinates on $\mathbb{A}^n$ and $(y_1 : \cdots : y_n)$.

(a) Describe the fibers of the projection $B\ell_0\mathbb{A}^n \to \mathbb{A}^n$.

(b) Describe the fibers of the projection $B\ell_0\mathbb{A}^n \to \mathbb{P}^{n-1}$.

(c) Let $Z \subset \mathbb{A}^2$ be $\{(x_1, x_2) : x_1x_2(x_1 - x_2) = 0\}$. Describe the subset of $B\ell_0$ lying above Z.

See the back for another fun problem!

Problem 6 Let G be an affine variety. Let $\mu : G \times G \to G$ be a regular map which makes G into a group. In this problem, we will show that G is isomorphic to a closed subgroup of GL_N.

Choose a basis e_i (probably infinite) for $\mathcal{O}_G$ as a k vector space. When we speak of g acting on $\mathcal{O}_G$, we mean $(g * v)(x) = v(xg)$ for g and $x \in G$ and $v \in \mathcal{O}_G$.

(a) Let u be a regular function on G and write $(\mu^* u)(g_1, g_2) = \sum_{i=1}^{N} v_i(g_1) e_i(g_2)$, where g_1 and g_2 are the first and second factors in $G \times G$. (See Problem 1 on the previous problem set.) Show that the vector space $\mathrm{Span}_k(v_1, \ldots, v_N)$ inside $\mathcal{O}_G$ is taken to itself by the action of G on $\mathcal{O}_G$, and that $u \in \mathrm{Span}_k(v_1, \ldots, v_N)$.

Let W be a finite dimensional subspace of $\mathcal{O}_G$ which is taken to itself by the G-action. Choose a new basis, f_i, for $\mathcal{O}_G$ so that $f_1, f_2, \ldots, f_N$ is a basis for W.

(b) For $i \leq N$, let $\mu^*(f_i)(g_1, g_2) = \sum_j f_j(g_1) a_{ij}(g_2)$. Show that a_{ij} is 0 for $j > N$. Show that the regular map $\rho : g \mapsto a_{ij}(g)$ from G to GL_N is a representation.

(c) Show that we can chose W as above so that W generates $\mathcal{O}_G$ as a k algebra. Show that $\rho(G)$ is closed in GL_N, and $\rho : G \to \rho(G)$ is an isomorphism.