Problem Set 6 – due October 22

See the course website for policy on collaboration.

Definitions/Notation Let $X = \operatorname{MaxSpec} A$ and $Y = \operatorname{MaxSpec} B$ be affine varieties and $\phi : Y \to X$ a regular map. We say that ϕ is ***finite*** if ϕ^* makes B into a finitely generated A-module.

Warning: Shavarevich implies that A must inject into B. We do NOT require this.

For X and Y quasi-projective varieties, and $\phi : Y \to X$ a regular map, ϕ is ***finite*** if X has a cover $\bigcup U_i$ by open affines for which each $\phi^{-1}(U_i)$ is affine, and $\phi^{-1}(U_i) \to U_i$ is finite. Although Shavarevich doesn't point this out, it is equivalent to ask that, for **every** open affine V of X, the preimage $\phi^{-1}(V)$ is affine and $\phi^{-1}(V) \to V$ is affine. See Proposition 8.21 in Milne for a proof.

Let K/k be an extension of fields. Elements t_1, t_2, ..., t_d in K are said to be ***algebraically independent*** if they do not obey any nonzero polynomial relation $f(t_1, t_2, \ldots, t_n) = 0$ where f has coefficients in k. Any two maximal algebraically independent sets have the same cardinality, this cardinality is called the ***transcendence degree*** of K/k. If A is a finitely generated k-algebra and is a domain, we define the dimension of $\operatorname{MaxSpec} A$ to be the transcendence degree of $\operatorname{Frac} A/k$. If X is an irreducible quasi-projective variety, then we define $\dim X$ to be $\dim U$ where U is any open affine subvariety of X; this definition does not depend on the choice of U (Problem 5).

Problem 1 Let X and Y be affine varieties, $\phi : Y \to X$ a regular map, and $q \in \mathcal{O}_X$.

(a) Show that $\phi^{-1}(D(q)) = D(\phi^* q)$.

(b) Show that, if $Y \to X$ is finite, then $D(\phi^* q) \to D(q)$ is finite as well.

Problem 2 Let $\phi : \mathbb{A}^1 \to \mathbb{A}^n$ be a regular map. Show $\phi(\mathbb{A}^1)$ is closed. (Hint: Use finiteness.)

Problem 3 In the proof of his second statement of Noether normalization (Theorem I.5.4.10), Shavarevich implicitly assumes the following result:

> Let $X \subsetneq \mathbb{A}^n$ be Zariski closed in $\mathbb{A}^n$. Let $\overline{X}$ be the closure of X in $\mathbb{P}^n$. Then there is some point of $\mathbb{P}^n \setminus \mathbb{A}^n$ which is not in $\overline{X}$.

Prove this. (Hint: First reduce to the case that X is defined by a single polynomial.)

Problem 4 Let X and Y be quasi-projective varieties and $\phi : Y \to X$ a regular map **with finite fibers**. In this problem, we will establish that the following are equivalent:

(1) ϕ is finite

(2) We can embed Y as a closed subset of $X \times \mathbb{P}^n$ so that ϕ is projection onto the X component.

We begin by showing (1) $\implies$ (2).

(a) Let U, V and P be affine varieties and $U \to V \times P$ a regular map. Suppose that the composition $U \to V \times P \to V$ is finite. Show that $U \to V \times P$ is finite.

(b) Remove the hypothesis that P be affine from the previous problem.

Choose an embedding $\iota : Y \subset \mathbb{P}^n$ and let map Y to $P \times X$ by $\alpha(y) = (\phi(y), \iota(y))$.

(c) Assume (1) and show that α is injective and has closed image. Conclude (2).

Now, assume (2). We will prove (1).

(d) Choose any $x \in X$. Show that there is a linear chart $L \subset \mathbb{P}^n$ with $Y \cap (\{x\} \times \mathbb{P}^n) \subset \{x\} \times L$.

(e) Show that there is an open neighborhood U of x so that $Y \cap (U \times \mathbb{P}^n) \subset U \times L$.

(f) In the above notation, show that $Y \cap (U \times \mathbb{P}^n) \longrightarrow U$ is finite. (Hint: Shavarevich Theorem I.5.3.7.) Explain why this shows (2) $\implies$ (1).

Problem 5.(a) Let X be a quasi-projective variety and let U and V be open affine subvarieties. Show that $U \cap V$ is affine. (This is surprisingly hard! Hint: Problem 4.(a) on Set 5 is relevant.)

(b) Let X be an irreducible quasi-projective variety and let $U = \operatorname{MaxSpec} A$ and $V = \operatorname{MaxSpec} B$ be open affine subvarieties. Show that $\operatorname{Frac} A \cong \operatorname{Frac} B$.

(c) Let $\phi : Y \to X$ be a regular map of quasi-projective varieties. Let $U \subset X$ and $V \subset Y$ be open affine subvarieties. Show that $\phi^{-1}(U) \cap V$ is affine.

Problem 6 Compute the dimensions of the following varieties:
(a) $\{(w, x, y, z) : wz - xy = 0\}$
(b) $\{(w, x, y, z) : wz - xy = w = x = 0\}$
(c) $\{(w, x, y, z) : wy = x^2,\ xz = y^2,\ wz = xy\}$.

Problem 7 Let A be a finitely generated reduced k-algebra and G a finite group acting on A by k-algebra automorphisms. Let $X = \operatorname{MaxSpec} A$, and we consider the corresponding action of G on X. Let A^G be the ring of G-invariant functions in A. In this problem, we will show that A is finitely generated and that it makes sense to define X/G to be $\operatorname{MaxSpec} A^G$.

Let $a_1, a_2, \ldots, a_s$ be a list of generators for A as a k-algebra. For $1 \le r \le s$ and $1 \le k \le |G|$, define e_{kr} by the following formula (t is a formal variable):

$$\prod_{g \in G} (t - g(a_r)) = t^{|G|} - e_{1r} t^{|G|-1} + e_{2r} t^{|G|-2} - \cdots \pm e_{|G|r}.$$

Let E be the sub-k-algebra of A generated by the e_{kr}.

(a) Show that $E \subseteq A^G$. Show that A is finitely generated as an E-module.

(b) Conclude that A^G is finitely generated as an E-module. (Hint: E is finitely generated, hence Noetherian.) Show that A^G is finitely generated as a k-algebra.

The first two parts of the problem permit us to talk about $\operatorname{MaxSpec} A^G$. Set $Y = \operatorname{MaxSpec} A^G$ and let $\pi : X \to Y$ be the map corresponding to $A^G \hookrightarrow A$. The remaining parts of this problem are largely independent of (a) and (b).

(c) Show that π is finite and surjective.

(d) Let $x \in X$ and $g \in G$. Show that $\pi(x) = \pi(gx)$.

(e) Let $x \in X$ and let $Gx = \{gx : g \in G\}$. Let y be a point of X not in Gx. Show that $\pi(x) \neq \pi(y)$. (Hint: First find an element $a \in A$ with $a(y) = 0$ and a nonzero at all the points of Gx, then modify a to be in A^G.)