Problem Set 7 – due October 29

See the course website for policy on collaboration.

Definitions/Notation For V a finite dimensional vector space, we write $G(d, V)$ (called the ***Grassmannian***) for the set of tensors in $\mathbb{P}(\bigwedge^d V)$ which are of the form $v_1 \wedge v_2 \wedge \cdots \wedge v_d$. On Friday, we will prove this is a closed subvariety of $\mathbb{P}(\bigwedge^d V)$. Moreover, there is a closed subvariety $\mathcal{S}$, called the ***tautological bundle***, of $G(d, V) \times V$ so that $\mathcal{S} \cap (\{\omega\} \times V)$ is a d-dimensional sub-space of V which we will denote $L(\omega)$. We can recover ω from $L(\omega)$ as follows: If $v_1, \ldots, v_d$ is a basis for $L(\omega)$ then $\omega = v_1 \wedge \cdots \wedge v_d$ as points of $\bigwedge^d V$.

Problem 1 Let $\pi : Y \to X$ be a regular map of affine varieties. Back on Problem Set 2, you saw that $\pi(Y)$ need be neither open nor closed.

A weaker notion than open or closed is "constructible". A subset Q of X is defined to be ***constructible*** if X can be built from finitely many open and closed sets, combined with the operations of $\cup$ and $\cap$. For example, if U_1 and U_2 are open, and Z_1, Z_2 and Z_3 are closed, then $\big((U_1 \cap Z_1) \cup (U_2 \cap Z_2)\big) \cap Z_3$ is constructible.

In this problem, we will prove

Chevalley's Theorem In the above notation, $\pi(Y)$ is constructible.

The proof is by induction on $\dim X$. We'll write C_d for the assertion that $\pi(Y)$ is constructible if $\dim X \leq d$.

(a) Show that, if C_d holds for π dominant (meaning that $X = \overline{\pi(Y)}$), then C_d holds.

(b) Show that, if C_d holds for X irreducible and π dominant, then C_d holds for π dominant.

(c) Show that, if C_{d-1} holds, then C_d holds for X irreducible and π dominant. (Hint: In our proof of semicontinuity of dimension, we established the following lemma: If X is irreducible and $\pi : Y \to X$ is dominant, there is a nonempty open set U contained in $\pi(Y)$.)

Problem 2 (This is a lemma we will need on Friday.) Let V be a finite dimensional vector space, ω an element of $\bigwedge^k V$, and $v_1, v_2, \ldots, v_r$ linearly independent vectors in V. Suppose that $v_i \wedge \omega = 0$, for $1 \leq i \leq r$. Show that we can write $\omega = v_1 \wedge v_2 \wedge \cdots \wedge v_r \wedge \eta$ for some $\eta \in \bigwedge^{k-r} V$.

Problem 3.(a) Let W be a subspace of V and r a positive integer. Show that $\{\omega : \dim L(\omega) \cap W \geq r\}$ is Zariski closed. (This is an example of a ***Schubert subvariety*** of $G(d, V)$.

(b) Let $\mathcal{F}\ell_n$ be $\{(\omega_1, \omega_2, \ldots, \omega_{n-1}) \in G(1, n) \times \cdots \times G(n-1, n) : L(\omega_1) \subset L(\omega_2) \subset \cdots \subset L(\omega_{n-1})\}$. Show that $\mathcal{F}\ell_n$ is closed in $G(1, n) \times \cdots \times G(n-1, n)$. ($\mathcal{F}\ell_n$ is called the ***flag variety***.)

Problem 4 Let C_3 be the vector space of homogenous cubic polynomials in w, x, y and z. Let $\mathcal{X} \subset G(2, 4) \times \mathbb{P}(C_3)$ be $\{(\omega, c) : c|_{L(\omega)} = 0\}$. In other words, I think of a point of $\mathcal{W}$ as a line and a cubic surface containing that line.

(a) Show that $\dim \mathbb{P}(C_3) = 19$ and $\dim G(2, 4) = 4$. (Hint for the latter: In class, we wrote down a dense open subset of $G(d, V)$ whose dimension is easy to compute.)

(b) Show that $\mathcal{X}$ is closed in $G(2, 4) \times \mathbb{P}(C_3)$.

(c) Show that every fiber of $\mathcal{X} \to G(2, 4)$ is isomorphic to $\mathbb{P}^{15}$. Conclude that $\dim \mathcal{X} = 19$.

(d) In parts (e) and (f), we will exhibit a cubic c which contains a finite, nonzero, number of lines. Using this, show that $\mathcal{X} \to \mathbb{P}(C_3)$ is surjective. So every cubic surface contains a line!

Let $Z(c_{\text{Fermat}}) = w^3 + x^3 + y^3 + z^3$, and assume k does not have characteristic 3.

(e) Find 27 lines in c_{Fermat}.

(f) Prove your list is complete. Hint for one approach: Suppose that the line through $(w_1 : x_1 : y_1 : z_1)$ and $(w_2 : x_2 : y_2 : z_2)$ lies in $Z(c_{\text{Fermat}})$. First show that

$$\det \begin{pmatrix} w_1^3 & x_1^3 & y_1^3 & z_1^3 \\ w_1^2 w_2 & x_1^2 x_2 & y_1^2 y_2 & z_1^2 z_2 \\ w_1 w_2^2 & x_1 x_2^2 & y_1 y_2^2 & z_1 z_2^2 \\ w_2^3 & x_2^3 & y_2^3 & z_2^3 \end{pmatrix} = 0.$$