Problem Set 8 – due November 5

See the course website for policy on collaboration.

Definitions/Notation Some more definitions regarding Grassmannians: If V has a chosen basis $e_1, \ldots, e_n$ and we have $v_1 \wedge \cdots \wedge v_d = \sum_{1<i_1<\cdots<i_d<n} p_{i_1 i_2 \cdots i_d} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d}$, then we call the $\binom{n}{d}$ numbers $p_{i_1 i_2 \cdots i_d}$ the ***Plücker coordinates*** of $\mathrm{Span}_k(v_1, \ldots, v_d)$. If we choose a decomposition of V as $A \oplus B$, where $\dim A = d$, then get embedding $s : \mathrm{Hom}(A, B) \hookrightarrow G(d, V)$, so that $L(s(f))$ is the graph of f. This is a dense open subset of $G(d, V)$, isomorphic to $\mathbb{A}^{d(\dim V - d)}$. We'll call this a ***Schubert chart*** (this is not standard terminology).

If I is a graded radical ideal in $k[x_1, \ldots, x_n]$, with corresponding projective variety $X \subset \mathbb{P}^{n-1}$, we define the ***Hilbert function*** of X by $h_X^{\mathrm{func}}(t) = \dim_k k[x_1, \ldots, x_n]_t / I_t$. We define the Hilbert polynomial $h_X^{\mathrm{poly}}(t)$ to be the unique polynomial such that $h_X^{\mathrm{poly}}(t) = h_X^{\mathrm{func}}(t)$ for $t \gg 0$. We showed that $h(t)$ has degree $\dim X$ and defined the ***degree*** of X is the integer $\deg X$ such that the leading term of $h(t)$ is $\frac{\deg X}{(\dim X)!} t^{\dim X}$.

Problem 1 Let W be the 2-dimensional subspace of k^5 spanned by the rows of

$$\begin{pmatrix} 1 & 0 & 2 & 3 & 5 \\ 0 & 1 & 7 & 11 & 13 \end{pmatrix}$$

(a) Compute the Plücker coordinates of W. (That is to say, the $\binom{5}{2}$ numbers p_{12}, p_{13}, p_{14}, p_{15}, p_{23}, p_{24}, p_{25}, p_{34}, p_{35}, p_{45}.)

(b) Let $W^\perp$ be $\{\vec{x} : \vec{w} \cdot \vec{x} = 0 \text{ for all } \vec{w} \in W\}$. (Here $\cdot$ is the dot product.) Find a basis of $W^\perp$.

(c) Compute the $\binom{5}{3}$ Plücker coordinates of $W^\perp$: (That is to say, the $\binom{5}{3}$ numbers p_{123}, p_{124}, p_{125}, p_{134}, p_{135}, p_{145}, p_{234}, p_{235}, p_{245}, p_{345}.)

Problem 2.(a) For $W \subset k^n$, let $W^\perp$ be $\{\vec{x} : \vec{w} \cdot \vec{x} = 0 \text{ for all } \vec{w} \in W\}$. (Again, $\cdot$ is the dot product.) Show that the correspondence $W \mapsto W^\perp$ is a regular map $G(d, k^n) \to G(n-d, k^n)$.

(b) Find a formula expressing the Plücker coordinates of $W^\perp$ in terms of the Plücker coordinates of W. (The formula I am thinking of is extremely explicit and fairly simple.)

Problem 3 Let $f(w, x, y, z)$ and $g(w, x, y, z)$ be relatively prime nonzero homogenous polynomials of degrees a and b. Compute the Hilbert polynomial of $Z(f, g)$ in $\mathbb{P}^3$.

Problem 4 Let X be $\mathbb{P}^1 \times \mathbb{P}^2$ embedded into $\mathbb{P}^5$ by the Segre map $(p : q) \times (r : s : t) \mapsto (pr : ps : pt : qr : qs : qt)$. Compute the Hilbert polynomial and degree of X in $\mathbb{P}^5$. Hint: It is easier and more useful to describe the quotient $k[x_1, x_2, x_3, x_4, x_5, x_6]/I$ than to give generators for I.

Problem 5 Let C be a conic in $\mathbb{P}^2$ and let p_1, p_2, p_3, p_4, p_5 and p_6 be 6 distinct points on C. We write L_{ij} for the line through p_i and p_j, and write λ_{ij} for a linear equation defining L_{ij}. In this problem we will use Bezout's theorem to prove:

Pascal's Theorem: The three points $L_{12} \cap L_{45}$, $L_{23} \cap L_{56}$ and $L_{34} \cap L_{16}$ are colinear.

(a) Show there is a constant $c \in k$ such that $\lambda_{12}\lambda_{34}\lambda_{56} + c\lambda_{23}\lambda_{34}\lambda_{56}$ vanishes at 7 points of C.

(b) Define $E = Z(\lambda_{12}\lambda_{34}\lambda_{56} + c\lambda_{23}\lambda_{34}\lambda_{56})$. Show that $E = C \cup M$ for some line M.

(c) Show that M passes through the points $L_{12} \cap L_{45}$, $L_{23} \cap L_{56}$ and $L_{34} \cap L_{16}$.

This was a short problem set. How about taking some of the extra time to read/think for your final paper?