Problem Set 9 – due November 12

See the course website for policy on collaboration.

Definitions/Notation Let V be a finite dimensional k vector space, X a Zariski closed subvariety of V and x a point of X. The ***Zariski tangent space to*** X ***at*** x, denoted $T_x X$, is the subspace of V consisting of those $\vec{v}$ such that $\frac{d}{dt} f(x + t\vec{v}) = 0$ for all $f \in I(X)$. It is enough to check this condition for a list of generators of $I(X)$. The ***Zariski cotangent space to*** X ***at*** x, denoted $T_x^* X$, is the dual vector space, a quotient of V^*. Letting $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_X$ at x, we have a natural isomorphism $\mathfrak{m}_x/\mathfrak{m}_x^2 \cong T_x^* X$.

We use this definition in nonreduced rings as well: For a commutative ring A and a maximal ideal $\mathfrak{m}$ of A, the Zariski cotangent space of A at $\mathfrak{m}$ is $\mathfrak{m}/\mathfrak{m}^2$, an $A/\mathfrak{m}$ vector space, and the Zariski tangent space is the $A/\mathfrak{m}$ dual.

For X irreducible, we say that X is ***nonsingular*** at x if $\dim T_x X = \dim X$. We also say X is ***regular***[1] or ***smooth*** at x. One can check that the property of being nonsingular at x is local, so we define a quasi-projective variety X to be nonsingular at x if some (equivalently any) open affine subset of X is nonsingular at x.

Let V be a finite dimensional k vector space and X a Zariski closed subvariety of V. We write A for the coordinate ring of X. The ***tangent bundle*** TX to X is the subvariety

$$\left\{ (x, \vec{v}) : f(x) = 0 \text{ and } \frac{d}{dt} f(x + t\vec{v}) = 0 \ \forall f \in I(X) \right\} \subset V \times V.$$

A ***vector field*** on X is a regular section: $\sigma : X \to TX$ of the projection $TX \to X$. The ***Kähler differentials*** Ω_X^1 are the regular functions on TX which are linear functions on each tangent space. Ω_X^1 is an A-module, generated by symbols df for all $f \in A$, with relations

$$d(u+v) = du + dv, \ d(uv) = udv + vdu, \ da = 0 \text{ for } a \in k.$$

The coordinate ring of TX can be described intrinsically as the A-algebra generated by symbols df with the same relations. We have $\Omega_X^1 \otimes_A A/\mathfrak{m}_x \cong T_x^* X$.

Problem 1 Check that the subvariety $\{(w : x : y : z) : wy = x^2, xz = y^2, wz = xy\}$ of $\mathbb{P}^3$ is nonsingular.

Problem 2 Compute the Zariski tangent spaces to the following subvarieties of $\mathbb{A}^2$ at $(0,0)$:

$$y = 0, \ y = x^2, \ y(y - x^2) = 0$$

Problem 3 Let $X = \operatorname{MaxSpec} A$ be an affine variety and x a point of X.

(a) Show that $T_x X$ is isomorphic (as a k-vector space) to the set of maps $D : A \to k$ such that $D(f+g) = D(f) + D(g)$, $D(fg) = f(x)D(g) + g(x)D(f)$ and $D(a) = 0$ for $a \in k$.

(b) Give a bijection between $T_x X$ and the set of k-algebra homomorphisms $A \to k[\epsilon]/\epsilon^2$ such that the composite $A \to k[\epsilon]/\epsilon^2 \to k$ is reduction modulo $\mathfrak{m}_x$.

(c) Describe the vector space structure on $T_x X$ directly in terms of homomorphisms $A \to k[\epsilon]/\epsilon^2$.

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[1] As far as I know, "nonsingular" and "smooth" are always synonyms. In the context of finitely generated algebras over a perfect field, they is also equivalent to "regular". In general, "regular" is defined in more settings. For example, $\mathbb{Z}$ is regular at a prime p, because $\mathbb{Z}$ has Krull dimension 1 and $(p\mathbb{Z})/(p\mathbb{Z})^2$ is a one dimensional $\mathbb{Z}/p\mathbb{Z}$ vector space, but one wouldn't say $\mathbb{Z}$ is smooth. Also, over a non-perfect field, "regular" and "smooth" are both words that make sense, but smooth is more restrictive. For example, let k be a field of characteristic p, and $t \in k$ a non-p-th power. Let $A = k[x]/(y^2 - x^p + t)$ and let $\mathfrak{m}$ be the ideal $\langle x^p - t, y \rangle$. Then A is regular at $\mathfrak{m}$ because $\mathfrak{m}/\mathfrak{m}^2$ is a one dimensional $A/\mathfrak{m}$ vector space and A has dimension 1 (Exercise: check this!). But A is not smooth at $\mathfrak{m}$, because $\frac{\partial}{\partial x}(y^2 - x^p + t)$ and $\frac{\partial}{\partial y}(y^2 - x^p + t)$ both vanish in the field $A/\mathfrak{m}$.

Problem 4 Let $X = \mathrm{MaxSpec}\, A$ be an irreducible affine variety of dimension d and x a point of X. Suppose that x is a *singular* point of x. Let f_1, f_2, …, f_d be regular functions on X vanishing at x. Show that the Zariski tangent space to $A/\langle f_1, f_2, \ldots, f_d \rangle$ at $\mathfrak{m}_x$ is nonzero. Conclude that $A/\langle f_1, f_2, \ldots, f_d \rangle$ is not reduced.

Problem 5 Let $X = \mathrm{MaxSpec}\, A$ and $Y = \mathrm{MaxSpec}\, B$ be affine algebraic varieties and $\phi : X \to Y$ a regular map.

(a) Let $x \in X$ and let $y = \phi(x)$. Show that $\phi^* \mathfrak{m}_y \subseteq \mathfrak{m}_x$ and $\phi^* \mathfrak{m}_y^2 \subseteq \mathfrak{m}_x^2$, so we get a linear map $\phi^* : T_y^* Y \to T_x^* X$. We define ϕ_* to be the dual map.

(b) Construct a regular map $\phi_* : TX \to TY$ which descends to $\phi_* : T_x X \to T_{\phi(x)} Y$ for every $x \in X$.

Problem 6 This problem reuses the notations $\mathcal{X}$, c_{Fermat}, and C_3 from Problem on Problem Set 7. Let L denote the line $\{(u : -u : v : -v)\}$ in $Z(c_{\mathrm{Fermat}})$. Let $[c_{\mathrm{Fermat}}]$ be the point of $\mathbb{P}(C_3)$ corresponding to the Fermat cubic and let $[L]$ be the point of $G(2, k^4)$ corresponding to L.

Check that the equations defining $\mathcal{X}$ cut out $[L]$ as a reduced point of $G(2, k^4)$. (This is also true for the other 26 lines on c_{Fermat}), I'm just trying to contain the computation.) Hint: I recommend first passing to a Schubert chart on $G(2, k^4)$ and the cotangent space.

~~**Problem 7** We only defined tangent bundles to affine varieties. The morally right way to define the tangent bundle to a projective variety is by gluing the tangent bundles to affine charts, but we aren't able to glue abstract varieties this term. This problem explores a morally wrong way.~~

~~Let W be a finite dimensional vector space. We will be working with subspaces of $\mathbb{P}(W \oplus \bigwedge^2 W)$, and will write $(w : \eta)$ for an element of this space, where $w \in W$ and $\eta \in \bigwedge^2 W$. Define~~

$$T\mathbb{P}(W) = \{(w : \eta) : w \neq 0,\ w \wedge \eta = 0\}.$$

~~Let $\pi : T\mathbb{P}(W) \to \mathbb{P}(W)$ be the map $(w : \eta) \to w$.~~

This problem is removed because it is actually false, not just morally wrong. (Or at least I can't prove it is right.) It definitely defines a vector bundle over $\mathbb{P}(W)$, but I don't think that bundle is isomorphic to $T\mathbb{P}(W)$. Given a vector $\vec{w} \in W$, the tangent space $T_{\vec{w}}\mathbb{P}(W)$ is canonically isomorphic to $\mathrm{Hom}(k\vec{w}, W/k\vec{w})$. Given $\bar{w} \in \mathbb{P}(W)$, we can lift $\bar{w}$ to a vector $\vec{w}$ and lift a map $\phi : k\vec{w} \to W/k\vec{w}$ to $\tilde{\phi} : k\vec{w} \to W$. I can then try to define an element of $\bigwedge^2 W$ by $\vec{w} \wedge \tilde{\phi}(\vec{w})$. The wedge product removes the ambiguity in choosing the lift $\tilde{\phi}$. But, if we rescale our choice of $\vec{w}$ by some scalar a, then $\vec{w} \wedge \tilde{\phi}(\vec{w})$ rescales by a^2, not a, so the class of $(\vec{w}, \vec{w} \wedge \phi(\vec{w}))$ in $\mathbb{P}(W \oplus \bigwedge^2 W)$ is not well defined.

We could fix this by introducing weighted projective spaces, where different coordinates can rescale by different powers. Or we could Veronese embed to $\mathrm{Sym}^2(W) \oplus \bigwedge^2 W$, sending $(\bar{w}, \phi)$ to $(\vec{w} \otimes \vec{w}, \vec{w} \wedge \tilde{\phi}(\vec{w}))$. But, at this point, what was already a hard problem has become completely out of hand.