POTENTIAL PAPER TOPICS

This course will require a final expository paper on some subject in algebraic geometry. I am imagining a paper of $8-15$ pages in length. The paper will be due Monday, December 8.

I'd love to talk to you about your interests and how to find paper topics that might fit them.

I also am glad to keep talking to you as you work on the paper.

Here are some ideas for potential topics:

• **Gröbner bases** How does a computer algebra system test whether an ideal contains a particular element? What about intersecting an ideal with a subring, computing the radical or the saturation of an ideal, or finding some particular point on $Z(I)$? All of these methods begin with Büchberger's algorithm and Gröbner bases. Learn how this works and how to perform a few computations of your choice?

• **Toric varieties** Toric varieties are a class of algebraic varieties which are both extremely combinatorial and serve as good examples for a broad range of purposes. How do we convert between combinatorial pictures with polytopes and algebraic equations?

◦ An awesome application of toric varieties is **Bernstein's theorem**: You know (or will by the end of the course) that a typical equation of degree d and a typical equation of degree e should have de common zeroes. But what if we know not just the degrees of the equations, but what sort of monomials they contain: How many common zeroes do $a + bx + cy$ and $dx + ey + fxy$ have, for generic (a, b, c, d, e, f)? There is an answer, which involves a bunch of clever polyhedral combinatorics, and some clever use of toric varieties.

• **Danielewski's surfaces** Danielewski solved a question posed by Zariski, constructing two algebraic surfaces X and Y such that $X \times \mathbb{C} \cong Y \times \mathbb{C}$ but $X \ncong Y$. This construction, and its generalizations by people who came after him, is ingenious but elementary. (You need some nontrivial topology to show that $X \ncong Y$, though.) Explain how this works.

• **Grassmannians, flag varieties etcetera** Grassmannians are projective varieties which parametrize the space of k planes in n space. (Just as projective spaces parametrize lines in n-space.) We'll talk about them some, but there is a ton more to say.

◦ What are the defining relations for their coordinate rings? How can you compute in these rings?

◦ What are the Schubert varieties? How do people use Schubert calculus to solve geometric problems?

◦ The Borel-Weil-Bott Theorem states (in part) that the irreducible representations of Lie groups are encoded in the coordinate rings of flag varieties and their relatives. In particular, the coordinate ring for the flag variety encodes the irreducible representations of SL_n. Explain how this works?

• **Goppa codes** Some of the best known error correcting codes are built using algebraic curves with many points over finite fields. How is this done?

• **Intersection theory on surfaces** We'll talk about intersecting curves in $\mathbb{P}^2$; there is a gorgeous theory on smooth projective surfaces in general. A great goal here would be to get to the Hodge Index Theorem.

• **Intersection theory** For a smooth projective variety X, the Chow ring $A(X)$ is a ring whose elements are equivalence classes of subvarieties of X and where multiplication, roughly, is intersection. Simply giving a precise definition of this multiplication is a paper in itself, would you like to write it?

• **Frobenius splitting** A ring with a Frobenius splitting is a commutative ring of characteristic p equipped with a map which acts like $\sqrt[p]{\ }$, in the sense that $\sqrt[p]{a+b} = \sqrt[p]{a} + \sqrt[p]{b}$, $\sqrt[p]{a^p b} = a\sqrt[p]{b}$ and $\sqrt[p]{1} = 1$. This extra structure highly constrains the geometry of the corresponding variety, and can be used to build nice stratifications of algebraic varieties. A nice example to work through might be to show how you can use Frobenius splitting to show that the radical ideal of the variety of matrices of rank $< k$ is generated by the $k \times k$ matrices.

• **Zeta functions of algebraic curves** Artin and Weyl built a beautiful analogy between number fields and algebraic curves over finite field, and defined the analogue of the ζ function. The functional equation turns into a short direct proof through Serre duality; the Riemann hypothesis has an elementary proof due to Bombieri. You'd probably want to already know at least a bit of the number theory story if you want to start on this.

• **Elliptic functions** The theory of elliptic curves was originally developed analytically, in terms of complex analytic functions with periodicity conditions. Explain why the ring of these periodic functions is the coordinate ring of an elliptic curve, and how to relate the algebraic group structure we construct to the group structure from analysis.

◦ **Multivariate Θ functions and abelian varieties** If periodic functions in one complex variable are too dull, you could write about periodic functions in many complex variables. This topic has a reputation for being difficult, but it is actually reasonably elementary, it just involves a lot of notation and analysis.

• **Constructing the Jacobian, algebraically** We'll study the group of line bundles on an algebraic curve as an abstract group. But this group also has the structure of an algebraic variety. Explain how that is constructed. This would probably be a better topic for people who have some familiarity with algebraic curves prior to this class.

• **Algebraic groups** I think we'd need to talk a bit about your background to figure out exactly what the scope of this paper is, since this is a field that fills books. But there are some great results: Every commutative algebraic group is the product of a finite groups, some number of copies of k^+ and some number of copies of $k^\times$. Every solvable algebraic group is a closed subgroup of the group of upper triangular matrices. For an algebraic group G and an element $g \in G$, we can uniquely write $g = g_{mult} g_{uni}$ where g_{mult} and g_{uni} commute, g_{mult} is diagonalizable in every representation and $g_{uni} - \mathrm{Id}$ is nilpotent in every representation.