An elementary Cech proof of the Riemann-Roch theorem

1. Introduction

The purpose of these notes is to give a proof of the Riemann-Roch theorem, for curves, in a language which will both appeal to students who have just finished a classical varieties course (out of, for example, Shafarevich's book) and will prepare them well for a course on sheaves and cohomology (for example, as taught from Hartshorne or Vakil's book). To the first end, I have made the following expositional choices:

- The proof focuses on concrete computations where possible.
- I use specific open sets, rather than local rings.
- I invoke Noether normalization and Hilbert series where appropriate.

To the second end, I have made the following choices:

- I use language which will extend well[1] to a course that discusses sheaf cohomology in depth, such as $H^0(\mathcal{O}(D))$ and $H^1(\mathcal{O}(D))$ rather than $L(D)$ and $R(D)$, etcetera.
- I give a Cech-style definition of $H^1(\mathcal{O}(D))$.
- I use commutative diagrams and long exact sequences wherever they are informative.

This proof is not original to me. It is the adelic proof due to Weil and popularized in Serre's *Algebraic Groups and Class Fields*, rewritten not to mention adeles. I have chosen to rewrite the proof because adeles, though beautiful and very important in number theory, are not central in modern algebraic geometry, and I think the introduction of adeles tends to make students think the proof is much more technical and abstract than it is.

I am grateful to Jake Levinson for reading an early draft of these notes. The presentation here is strongly influenced by Serre's presentation in Chapter II of *Algebraic Groups and Class Fields*; I also consulted Ravi Vakil's notes `math.stanford.edu/∼vakil/725/bagsrr.pdf` and Dolgachev's Lecture 17 at `http://www.math.lsa.umich.edu/∼idolga/631.pdf`.

2. Terminology

Throughout these notes, X denotes an irreducible smooth projective curve over an algebraically closed field k. For an open subset U of X, we write $\mathcal{O}_U$ for the regular functions on U. We write $\mathrm{Frac}(X)$ for the field of meromorphic functions on X, this is $\mathrm{Frac}\,\mathcal{O}_U$ for any nonempty affine open $U \subset X$. Similarly, we write Ω_U for the regular 1-forms on U and we write $\mathrm{Frac}\,\Omega(X)$ for the meromorphic 1-forms. (This last notation is non-standard.) $\mathrm{Frac}\,\Omega(X)$ is a one dimensional $\mathrm{Frac}(X)$ vector space, and is equal to $\Omega_U \otimes_{\mathcal{O}_U} \mathrm{Frac}(X)$ for any nonempty affine open U in X.

For every point $x \in X$, we write $\mathcal{O}_{X,x}$ for the local ring at x; this is a dvr. We write $\mathfrak{m}_x$ for the maximal ideal of $\mathcal{O}_{X,x}$. We write v_x for the valuation $\mathrm{Frac}(X)^\times \to \mathbb{Z}$ which takes the order of vanishing at x and define $v_x(0) = \infty$. So $\mathcal{O}_{X,x} = \{f \in \mathrm{Frac}(X) : v_x(f) \geq 0\}$. We also define valuations on $\mathrm{Frac}\,\Omega(X)$: If $x \in X$ and ω is an nonzero element of $\mathrm{Frac}\,\Omega(X)$, choose a uniformizer u_x at X. (Meaning that u_x generates $\mathfrak{m}_x$ or, equivalently, that $v_x(u_x) = 1$.) Write $\omega = f du_x$ for some $f \in \mathrm{Frac}(X)$. Then we define $v_x(\omega)$ to be $v_x(f)$; it is easy to check this is independent of the choice of u_x.

A ***divisor*** on X is a finite formal sum of points of X. We will write a divisor as $\sum_{x \in X} D(x)x$, where $D(x) \in \mathbb{Z}$ and $D(x)$ is 0 for all but finitely many x. The ***degree*** of a divisor, $\deg D$ is $\sum_{x \in X} D(x)$. Divisors form an abelian group, which we write additively. For two divisors D and E, we write $D \geq E$ to indicate that $D(x) \geq E(x)$ for all x.

[1] One way in which we are inconsistent with Hartshorne or Vakil's terminology is that the regular functions on an affine open U are denoted by $\mathcal{O}_U$, not $\mathcal{O}(U)$, so that we can distinguish this from $\mathcal{O}(D)$ without remembering what the symbol inside the parentheses denotes. Following this convention, I write $\mathcal{O}(D)_U$, not $\mathcal{O}(D)(U)$ for sections of $\mathcal{O}(D)$ over U.

For $f \in \operatorname{Frac}(X) \setminus \{0\}$ we define $\operatorname{div}(f)$ to be the divisor $\sum_{x \in X} v_x(f) x$. A divisor of the form $\operatorname{div}(f)$ is called ***principal***. Two divisors D and E are called ***rationally equivalent*** if $D - E$ is principal; we write $D \sim E$. For $\omega \in \operatorname{Frac}\Omega(X) \setminus \{0\}$, we define $\operatorname{div}(\omega)$ to be the divisor $\sum_{x \in X} v_x(\omega)$. If ω and η are two nonzero meromorphic 1-forms, then $\omega = f\eta$ for some $f \in \operatorname{Frac}(X) \setminus \{0\}$, and thus $\operatorname{div}(\omega) = \operatorname{div}(f) + \operatorname{div}(\eta)$ and $\operatorname{div}(\omega) \sim \operatorname{div}(\eta)$. A divisor of the form $\operatorname{div}(\omega)$ is called ***canonical***. We write K to mean an arbitrary canonical divisor; the symbol K will only occur in formulas which depend only on the rational equivalence class of the divisor being discussed.

For a divisor D, we define

$$\begin{array}{rcl} H^0(\mathcal{O}(D)) & = & \{f \in \operatorname{Frac}(X) : \operatorname{div}(f) + D \geq 0\} \\ H^0(\Omega(D)) & = & \{\omega \in \operatorname{Frac}\Omega(X) : \operatorname{div}(\omega) + D \geq 0\} \end{array}.$$

Note that, as D gets more positive, these vector spaces grow larger.

3. The Riemann-Roch Theorem

Our main result is:

The Riemann-Roch Theorem. *There is a non-negative integer g, called the genus of X, for which*

$$\dim H^0(\mathcal{O}(D)) - \dim H^0(\Omega(-D)) = \deg D - g + 1.$$

We can also write this as

$$\dim H^0(\mathcal{O}(D)) - \dim H^0(\mathcal{O}(K - D)) = \deg D - g + 1.$$

We observe that this theorem implies two concrete descriptions of g:

Proposition 1. *The integer g can be defined by either of the formulas:*

$$\dim H^0(\Omega) = g \text{ or } \deg(K) = 2g - 2.$$

Proof. Taking $D = 0$, we have $\dim H^0(\mathcal{O}) - \dim H^0(\Omega) = 1 - g$, so $\dim H^0(\Omega) = g$. Adding together the second equation for D and for $K - D$, we get $0 = \deg D + \deg(K - D) - 2g + 2$ so $\deg K = 2g - 2$. □

In particular, we see from the first formula that $g \geq 0$.

4. Approximate Riemann Roch

The aim of this section is to prove the following result, which says that the Riemann-Roch theorem is correct up to constant error, and point out its corollaries and variants. This part of the Theorem is due to Riemann by himself.

Approximate Riemann-Roch. *We have*

$$\dim H^0(\mathcal{O}(D)) = \max(\deg D, 0) + O(1)$$

where the constant in the $O(1)$ depends only on the curve X and not the divisor D. If $\deg D < 0$, then $\dim H^0(\mathcal{O}(D)) = 0$, and there are constants M and h (dependent only on X) so that $\dim H^0(\mathcal{O}(D)) = \deg(D) + h$ whenever $\deg D > M$.

The claim about $H^0(\mathcal{O}(D))$ for $\deg D < 0$ is obvious; we include it for symmetry with the highly nonobvious claim about $H^0(\mathcal{O}(D))$ for $\deg D \gg 0$.

We begin with the straightforward observation:

Proposition 2. *If $D' = D + p$ for some point p of X, then $H^0(\mathcal{O}(D))$ is a subspace of $H^0(\mathcal{O}(D'))$ and*

$$\dim H^0(\mathcal{O}(D'))/H^0(\mathcal{O}(D)) = (0 \text{ or } 1).$$

We immediately deduce the corollary

Proposition 3. *If $D' \geq D$, then*

$$\dim H^0(\mathcal{O}(D)) \leq \dim H^0(\mathcal{O}(D')) \leq \dim H^0(\mathcal{O}(D)) + \deg(D' - D).$$

In particular, if $\deg D < 0$, then $H^0(\mathcal{O}(D)) = 0$ (since any nonzero rational function f has $\deg \operatorname{div}(f) = 0$.) So Proposition 3 implies

$$\dim H^0(\mathcal{O}(D')) \leq \deg D' + 1$$

for any D'. We have proved the upper bound in Approximate Riemann-Roch; the rest of this section proves the lower bound.

Embed X into some projective space. Fix a Noether normalization $\pi : X \to \mathbb{P}^1$ of degree n and let U be the open set $X \setminus \pi^{-1}(\infty)$. We note that U is the intersection of X with an affine chart in $\mathbb{P}^N$, so U is affine. Let D_∞ be the divisor which is supported on the points $\pi^{-1}(\infty)$ and where $x \in \pi^{-1}(\infty)$ is counted with multiplicity equal to the ramification of π at x. Let the embedding $X \hookrightarrow \mathbb{P}^N$ have degree n, so π has degree n and $\deg D_\infty = \deg \pi = n$.

Our first result is that Approximate Riemann-Roch holds when D is a multiple of D_∞.

Proposition 4. *We have*

$$\dim H^0(\mathcal{O}(tD_\infty)) = tn + O(1) \text{ for } t \geq 0.$$

Moreover, there is a constant h such that

$$\dim H^0(\mathcal{O}(tD_\infty)) = tn + h \text{ for } t \gg 0.$$

Here the constant h, and the implicit constant in the $O(1)$, depend on X and the choice of divisor D_∞, but not on t.

Remark. Morally, the right proof is that $H^0(\mathcal{O}(tD_\infty))$ is roughly the degree t part of the homogenous coordinate ring of X, so this should follow from the result that Hilbert polynomials exist and have the stated leading term. However, $H^0(\mathcal{O}(tD_\infty))$ is only the degree t part of the homogenous coordinate ring for t sufficiently large, and we don't have the tools to prove that fact efficiently. Therefore, we give a direct proof of the proposition instead.

Proof. As pointed out above, we have the easy bound $\dim H^0(\mathcal{O}(tD_\infty)) \leq \deg(tD_\infty) + 1 = tn + 1$. So our goal is to prove a lower bound. Notice that $\bigcup_{t \geq 0} H^0(\mathcal{O}(tD_\infty)) = \mathcal{O}_U$.

Write $k[z]$ for the regular functions on $\mathbb{P}^1 \setminus \{\infty\}$. The Noether normalization π makes $\mathcal{O}_U$ into a finite $k[z]$-module of rank n. Let $f_1, f_2, \ldots, f_n$ be elements of $\mathcal{O}_U$, linearly independent over $\operatorname{Frac} k[z]$. (Since all finitely generated torsion free modules over a PID are free, we could choose f_i to be a $k[z]$ basis of $\mathcal{O}_U$, but this isn't necessary for our present purposes.) Choose s large enough that all the f_i lie in $H^0(\mathcal{O}(sD_\infty))$. Then, for $t \geq s$, all the functions

$$\begin{gathered} f_1, zf_1, \ldots, z^{t-s}f_1, \\ f_2, zf_2, \ldots, z^{t-s}f_2, \\ \vdots \\ f_n, zf_n, \ldots, z^{t-s}f_n \end{gathered}$$

are in $H^0(\mathcal{O}(tD_\infty))$ and are linearly independent over k. So

$$\dim H^0(\mathcal{O}(tD_\infty)) \geq (t - s + 1)n = tn - O(1).$$

This proves the first claim.

For the second claim, note that Proposition (2) implies that $\dim H^0(\mathcal{O}(tD_\infty)) - tn$ is a weakly decreasing function of t. A weakly decreasing bounded integer function is eventually constant. $\square$

We now prove the following consequence of Proposition 4:

Proposition 5. *There is an absolute constant C (dependent only on the curve X) such that any divisor D is rationally equivalent to a divisor of the form $tD_\infty + E$ where $\sum_{x \in X} |E(x)| \leq C$.*

Remark. We can think of this as saying that the group of divisors modulo rational equivalence is roughly the product of $\mathbb{Z}$ and something finite dimensional. This can be made very precise: Write $\mathrm{Pic}(X)$ for the group of divisors modulo rational equivalence. We have a short exact sequence $0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \stackrel{\deg}{\longrightarrow} \mathbb{Z} \to 0$, where $\mathrm{Pic}^0(X)$ is defined as the kernel of the degree map. (This sequence is non-canonically split, because any sequence of abelian groups of the form $0 \to A \to B \to \mathbb{Z} \to 0$ is split.) There is a g-dimensional abelian algebraic group[2] J called the ***Jacobian*** of X such that, as abstract groups, $J \cong \mathrm{Pic}^0(X)$. The Jacobian was implicit in the work of Abel-Jacobi in the complex case and was first constructed as a complex algebraic variety by Riemann. Weil reinvented the entire foundations of algebraic geometry in order to build the Jacobian over a general field. Even with modern tools, it is still quite challenging. My favorite reference for the modern approach is Kleiman's article *The Picard Scheme* in *Fundamental algebraic geometry: Grothendiecks FGA explained.*

Proof. We can write any divisor D as $D_+ - D_-$, where D_+ and D_- are ≥ 0. So it suffices to prove the claim in the case that $D \geq 0$, and we assume this from now on. (Our constant C for general D will be twice the C which we obtain under the hypothesis $D \geq 0$.)

Moreover, we can decompose D as $D' + D''$ where D' is supported on U and D'' is supported on $\pi^{-1}(\infty)$. It is enough to prove the claim for D' and D''.

Let B be the constant for which $\dim H^0(tD_\infty) \geq tn - B$ (using Proposition 4). Choose[3] $t = \lceil (\deg D' + B + 1)/n \rceil$ so that $tn - B > \deg D'$. Note that $\deg D' = \dim \bigoplus_{x \in U} \mathcal{O}_{X,x}/\mathfrak{m}_x^{D'(x)}$, so we have forced $\dim H^0(tD_\infty)$ to be strictly greater than $\dim \bigoplus_{x \in U} \mathcal{O}_{X,x}/\mathfrak{m}_x^{D'(x)}$. So there is a nonzero function $f \in H^0(\mathcal{O}(tD_\infty))$ which reduces to 0 modulo $\mathfrak{m}_x^{D'(x)}$ for all $x \in U$. In other words, $\mathrm{div}(f) - D'$ is positive on U. Therefore, $\mathrm{div}(f) - D' + tD_\infty \geq 0$ everywhere.

Write $D' - \mathrm{div}(f) = tD_\infty + E'$. Then $D' \sim tD_\infty + E'$ and we have shown that $E' \leq 0$. But we also have

$$\deg E' = \deg D' - nt = \deg D' - n\lceil (\deg D' + B + 1)/n \rceil \geq -n(B+2).$$

So we have proven that D' can be approximated in the required way with $C = n(B+2)$.

We now must deal with D''. Let $D_0 = \pi^{-1}(0)$, again computed with multiplicity. Repeating the previous arguments shows that there is an absolute constant C'' such that D'' is rationally equivalent to a divisor of the form $tD_0 + E''$ with $\sum_{x \in X} |E''(x)| \leq C''$. But $D_0 \sim D_\infty$, so this proves the result. ☐

We now prove Approximate Riemann-Roch.

Proof. For any divisor D, we can have $D \sim tD_\infty + E$ where $\sum_{x \in X} |E(x)|$ is bounded by an absolute constant C. Since $\dim H^0(\mathcal{O}(D))$ depends only on the rational equivalence class of D, it is enough to prove the result for divisors of the form $tD_\infty + E$.

The result for tD_∞ is Proposition 4. Applying Proposition 3 twice, we see that changing from $D = tD_\infty$ to $D = tD_\infty + E$ can only change $\dim H^0(\mathcal{O}(D)) - \deg D$ by $\sum_{x \in X} |E(x)| \leq C$. ☐

We prove one corollary that we will need in the next section:

Proposition 6. *Let z be a point of X and let $x_1, x_2, \ldots, x_N$ be finitely many other points of X. For each x_i, let g_i be an element of* $\mathrm{Frac}(X)$ *and let d_i be an integer. Then there is an element $f \in \mathrm{Frac}(X)$ whose only poles are in $\{z, x_1, x_2, \ldots, x_N\}$ and such that $v_{x_i}(f - g_i) \geq d_i$.*

---

[2]There is a common convention in the theory of algebraic groups that an algebraic group is by definition affine. J is not affine, but rather projective. J is an algebraic group in the sense that it is an algebraic variety and there is a regular map $\mu : J \times J \to J$ which gives J the structure of an abelian group.

[3]The notation $\lceil x \rceil$ means to round x up.

Proof. Let $e_i = v_{x_i}(g_i)$. The conditions on f become more restrictive as d_i increases, so we may as well replace d_i by $\max(d_i, e_i)$, so we can assume that $d_i \geq e_i$ for all i. Our proof will be by induction on $\sum(d_i - e_i)$.

If $d_i = e_i$ for all i, then we may take $f = 0$. So assume that $d_i > e_i$ for some i; without loss of generality, take $d_N > e_N$. Let E be the divisor $\sum_i e_i x_i$ and let $E' = E - x_N$. If we choose t large enough, then Approximate Riemann-Roch shows that $\dim H^0(\mathcal{O}(tz+E)) = \dim H^0(\mathcal{O}(tz+E'))+1$. So there is a function $h \in \dim H^0(\mathcal{O}(tz+E))$ with a pole of order precisely $-e_N$ at x_N. We can choose a scalar c such that the leading terms of g_N and ch match. Replacing g_i by $g_i' := g_i - ch$ and E by E', we can inductively find $f' \in H^0(\mathcal{O}(tz+E'))$ with $v_{x_i}(f' - g_i + ch) \geq d_i$ for all i. Then take $f = f' + ch$.

□

5. Cohomology

For W any nonempty open subset of X, we define

$$\mathcal{O}(D)_W = \{f \in \operatorname{Frac}(X) : v_x(f) + D(x) \geq 0 \ \forall_{x \in W}\}.$$

So $H^0(\mathcal{O}(D)) = \mathcal{O}(D)_X$. We define $\Omega(D)_W$ similarly.

Let $X = U \cup V$ for opens U and V, with U and V neither equal to $\emptyset$ nor X. Then $H^0(\mathcal{O}(D)) = \mathcal{O}(D)_U \cap \mathcal{O}(D)_V$, where the intersection can be viewed as taking place either in $\operatorname{Frac}(X)$ or, more usefully for the moment, in $\mathcal{O}(D)_{U\cap V}$. In other words, we have an exact sequence:

$$0 \to H^0(\mathcal{O}(D)) \to \mathcal{O}(D)_U \oplus \mathcal{O}(D)_V \xrightarrow{\partial} \mathcal{O}(D)_{U\cap V}$$

where $\partial(f, g) = f - g$. We define $H^1(\mathcal{O}(D); U, V)$ to be the cokernel of ∂, so we have an exact sequence

$$0 \to H^0(\mathcal{O}(D)) \to \mathcal{O}(D)_U \oplus \mathcal{O}(D)_V \xrightarrow{\partial} \mathcal{O}(D)_{U\cap V} \to H^1(\mathcal{O}(D); U, V) \to 0 \quad (*)$$

$H^0(\mathcal{O}(D))$ is a globally defined concept, although we might use an open cover to compute it. The following lemma tells us that the same is true of H^1.

Proposition 7. *Let $X = U \cup V$ be an open cover with U, $V \neq \emptyset$, X. Let p be a point of $U \cap V$ and let $U' = U \setminus p$. Then the obvious map $H^1(\mathcal{O}(D); U, V) \to H^1(\mathcal{O}(D); U', V)$ is an isomorphism.*

By the obvious map, we mean the map on cokernels induced by the restriction maps $\mathcal{O}(D)_U \to \mathcal{O}(D)_{U'}$ and $\mathcal{O}(D)_{U\cap V} \to \mathcal{O}(D)_{U'\cap V}$.

Repeatedly using Proposition 7 gives us natural isomorphisms between $H^1(\mathcal{O}(D); U_1, V_1)$ and $H^1(\mathcal{O}(D); U_2, V_2)$ for any open covers (U_1, V_1) and (U_2, V_2) as above. We will therefore refer to $H^1(\mathcal{O}(D))$ without regard to a choice of cover.

Remark. If one traces through the isomorphisms carefully, one finds that the natural isomorphism $H^1(\mathcal{O}(D); U, V) \to H^1(\mathcal{O}(D); V, U)$ sends the class of $f \in \mathcal{O}(D)_{U\cap V}$ to the class of $-f \in \mathcal{O}(D)_{V\cap U}$.

Proof. Since there is at least one point in $X \setminus U$, Proposition 6 tells us that there are regular functions on U', and on $U' \cap V$, whose Laurent series at p starts with any given specified terms. Writing u_p for a uniformizer at p, we deduce that

$$\mathcal{O}(D)_{U'}/\mathcal{O}(D)_U \cong \mathcal{O}(D)_{U'\cap V}/\mathcal{O}(D)_{U\cap V} \cong \operatorname{Frac}(X)/u_p^{D(p)}\mathcal{O}_{X,p} \quad (\dagger)$$

We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}(D)_U \oplus \mathcal{O}(D)_V & \longrightarrow & \mathcal{O}(D)_{U'} \oplus \mathcal{O}(D)_V & \longrightarrow & \mathcal{O}(D)_{U'}/\mathcal{O}(D)_U & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(D)_{U\cap V} & \longrightarrow & \mathcal{O}(D)_{U'\cap V} & \longrightarrow & \mathcal{O}(D)_{U'\cap V}/\mathcal{O}(D)_{U\cap V} & \longrightarrow & 0
\end{array}$$

From equation (†), the third vertical map is an isomorphism.

Hence, by the snake lemma, we have a long exact sequence

$$0 \to H^0(\mathcal{O}(D)) \to H^0(\mathcal{O}(D)) \to 0 \to H^1(\mathcal{O}(D); U, V) \to H^1(\mathcal{O}(D); U', V) \to 0 \to 0. \qquad \square$$

Now that we know H^1 is well defined, we begin investigating its key properties.

Proposition 8. *As a k-vector space, $H^1(\mathcal{O}(D))$ is finite dimensional.*

Proof. Take a Noether normalization $\pi : X \to \mathbb{P}^1$ as in the preceding section. We will compute H^1 with respect to the open cover $U = \pi^{-1}(\mathbb{P}^1 \setminus \{\infty\})$ and $V = \pi^{-1}(\mathbb{P}^1 \setminus \{0\})$. Then $\mathcal{O}(D)_{U\cap V}$ is a free $k[z, z^{-1}]$-module of rank n and $\mathcal{O}(D)_U$ and $\mathcal{O}(D)_V$ are $k[z]$ and $k[z^{-1}]$ submodules of rank n. Identifying $\mathcal{O}(D)_{U\cap V}$ with $k[z, z^{-1}]^{\oplus n}$, we have $\mathcal{O}(D)_U \supset z^M k[z]$ and $\mathcal{O}(D)_V \supset z^{-M} k[z^{-1}]$ for some M, so $\dim H^1(\mathcal{O}(D)) \leq n(2M + 1)$. $\square$

Proposition 9. *If D and E are rationally equivalent, then $H^1(\mathcal{O}(D)) \cong H^1(\mathcal{O}(E))$. The isomorphism is natural once we choose a rational function f with divisor $D - E$.*

Proof. Let $E = D + \operatorname{div}(f)$. Multiplication by f induces compatible maps $\mathcal{O}(D)_W \to \mathcal{O}(E)_W$ for all open sets W and, in particular, for $W = U$, V and $U \cap V$. So multiplication by f induces a map on the cokernel in equation $(*)$ $\square$

The next proposition will be key in our future arguments, and in much analysis of H^0 and H^1.

Proposition 10. *Let D be a divisor, p a point of X and $D' = D + p$. Then we have a long exact sequence:*

$$0 \to H^0(\mathcal{O}(D)) \to H^0(\mathcal{O}(D')) \to k \to H^1(\mathcal{O}(D)) \to H^1(\mathcal{O}(D')) \to 0.$$

Remark. The 1-dimensional k vector space in the middle of the sequence is canonically $(T_pX)^{\otimes D(p)+1}$.

Proof. Since H^1 is defined independent of the choice of cover, we may assume that $p \in U \setminus V$. So $\mathcal{O}(D')_U/\mathcal{O}(D)_U \cong k$ and $\mathcal{O}(D)_V \cong \mathcal{O}(D')_V$ and $\mathcal{O}(D)_{U\cap V} \cong \mathcal{O}(D')_{U\cap V}$. We can piece these together into a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}(D)_U \oplus \mathcal{O}(D)_V & \longrightarrow & \mathcal{O}(D')_U \oplus \mathcal{O}(D')_V & \longrightarrow & k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(D)_{U\cap V} & \longrightarrow & \mathcal{O}(D')_{U\cap V} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Applying the snake lemma gives the conclusion. $\square$

Remark. A good exercise is to repeat this proof on the hypothesis that $p \in U \cap V$, and thus remove the use of Proposition 7.

We can now prove:

Homological Riemann-Roch. *There is a non-negative integer g for which*

$$\dim H^0(\mathcal{O}(D)) - \dim H^1(\mathcal{O}(D)) = \deg D - g + 1.$$

Proof. The equation makes sense by Proposition 8. Take $g = \dim H^1(\mathcal{O})$. Then the equality holds for $\mathcal{O}$, and Proposition 10 shows that both sides increase by 1 if we change D to $D + p$, so the result holds for any D. $\square$

Remark. If one has two vector spaces V and W and a map $\phi : V \to W$, it is usually easier to prove a theorem about $\dim \operatorname{Ker}(\phi) - \dim \operatorname{CoKer}(\phi)$ then to study $\operatorname{Ker}(\phi)$ or $\operatorname{CoKer}(\phi)$ separately. This is obviously true if V and W are finite dimensional, as $\dim \operatorname{Ker}(\phi) - \dim \operatorname{CoKer}(\phi) = \dim V - \dim W$, but it is a good guideline when V and W are infinite dimensional as well.

Combining Approximate Riemann Roch and Homological Riemann Roch, we deduce:

Proposition 11. *We have*

$$\dim H^1(\mathcal{O}(D)) = \max(-\deg D, 0) + O(1)$$

where the constant in the $O(1)$ depends only on the curve X and not the divisor D.

We will now be done if we can prove $\dim H^1(\mathcal{O}(D)) = \dim H^0(\Omega(-D))$. This will be the goal of the following sections.

6. The Serre duality pairing

The Serre duality theorem states that the vector spaces $H^1(\mathcal{O}(D))$ and $H^0(\Omega(-D))$ are naturally dual. (And thus, replacing D by $K-D$, that $H^0(\mathcal{O}(D))$ and $H^1(\Omega(-D))$ are naturally dual.) In this section, we will define a bilinear pairing $\langle\ ,\rangle : H^1(\mathcal{O}(D)) \times H^0(\Omega(-D)) \to k$. In the next section, we will prove that it is a perfect pairing.

To begin with, consider the case of $H^1(\Omega)$, which should be dual to the one dimensional vector space $H^0(\mathcal{O})$. We will not be able to prove that $\dim H^1(\Omega) = 1$ until the same moment that we prove Serre duality in full, but we can construct a linear map $\int : H^1(\Omega) \to k$. In order to describe this linear map, we first review the properties of residues of 1-forms on smooth curves.

Let x be a point of X and choose a uniformizer u in $\mathcal{O}_{X,x}$. Let ω be a rational 1-form on X. Then we can write ω in the form

$$\omega = \Big(\frac{a_{-N}}{u^N} + \frac{a_{-N+1}}{u^{N-1}} + \cdots + \frac{a_1}{u} + f \Big)\, du$$

for a_i various elements of the ground field and $f \in \mathcal{O}_{X,x}$. We define the ***residue*** of ω at x, denoted $\mathrm{res}_x(\omega)$, to be a_{-1}.

We will need the following properties of residues, whose proofs are surprisingly difficult.

Proposition 12. *The residue of ω at x does not depend on the choice of uniformizer u.*

Proposition 13. *For any $\omega \in \mathrm{Frac}\,\Omega(X)$, we have $\sum_{x\in X} \mathrm{res}_x \omega = 0$. This sum makes sense because all but finitely many terms are zero.*

Accepting these results for now, we will define a map $\int : H^1(\Omega) \to k$. Let (U,V) be an open cover of X and let ω be a 1-form on $U \cap V$, so Ω represents a class $[\omega]$ in $H^1(\Omega; U, V)$. Define

$$\int \omega = \sum_{x \in X\setminus U} \mathrm{res}_x \omega.$$

In order for this definition to make sense, we need to check two things:

Proposition 14. *This definition depends only on the class of ω in $H^1(\Omega; U, V)$.*

Proof. In other words, we must check that $\int \alpha - \beta$, where $\alpha \in \Omega_U$ and $\beta \in \Omega_V$, is 0. It is clearly enough to check that $\int \alpha$ and $\int \beta$ are 0. For $\alpha \in \Omega_U$, all the poles of α are in $X \setminus U$, so $\sum_{x\in X\setminus U} \mathrm{res}_x \omega = \sum_{x \in X} \mathrm{res}_x \omega = 0$ by Proposition 13. For $\beta \in \Omega_V$, there are no poles of β in $X \setminus U$ (since $X \setminus U \subset V$) so the sum is obviously zero. □

Proposition 15. *If (U,V) and (U',V') are two open covers of X, with $U' \subset U$ and $V' \subset V$, and ω is a 1-form on $U\cap V$, then $\int \omega$ is the same whether computed with respect to (U,V) or (U',V').*

Proof. The 1-form ω doesn't have any poles at the points of $(X \setminus U') \setminus (X \setminus U)$. □

Now, if $f \in H^0(\mathcal{O}(D))$ and $\omega \in \Omega(-D)_{U\cap V}$, then $f\omega$ is clearly in $\Omega_{U\cap V}$. We define the pairing $\langle\ ,\ \rangle : H^0(\mathcal{O}(D)) \times H^1(\Omega(-D)) \to k$ by $\langle f, [\omega]\rangle = \int f\omega$. Similarly, if $f \in \mathcal{O}(D)_{U\cap V}$ and $\omega \in \Omega(-D)_{U\cap V}$, then $\langle [f], \omega\rangle = \int f\omega$ defines a pairing between $H^1(\mathcal{O}(D))$ and $H^0(\Omega(-D))$. We can now state:

Serre Duality. *The pairings* $\langle\ ,\ \rangle : H^0(\mathcal{O}(D)) \times H^1(\Omega(-D)) \to k$ *and* $\langle\ ,\ \rangle : H^1(\mathcal{O}(D)) \times H^0(\Omega(-D)) \to k$ *are perfect pairings. As a corollary,* $\dim H^0(\mathcal{O}(D)) = \dim H^1(\Omega(-D)) = \dim H^1(\mathcal{O}(K-D))$ *and* $\dim H^1(\mathcal{O}(D)) = \dim H^0(\Omega(-D)) = \dim H^0(\mathcal{O}(K-D))$.

We prove Serre duality in the next section. For now, we discuss Propositions 12 and 13.

Analytic proof of Proposition 12. Consider X as a complex manifold and let γ be a little circle around x. Then $\mathrm{res}_x\omega = \frac{1}{2\pi i}\oint_\gamma \omega$. □

Analytic proof of Proposition 13. Let $x_1, x_2, \ldots, x_r$ be the points of X where ω has poles. Let D_i be a little disc around x_i. We want to show that $\sum_i \frac{1}{2\pi i}\oint_{\partial D_i}\omega = 0$. Defining $U = X \setminus \bigcup D_i$, we want to show that $\oint_{\partial U}\omega = 0$. Since ω is holomorphic on U, this is true. If your complex analysis course only proved Cauchy's integral theorem in $\mathbb{C}$ rather than on a Riemann surface, we can also use Stokes' theorem: If f is analytic then $d(f(z)dz) = 0$, so $d\omega = 0$. □

The remainder of this section aims to remove the analysis from these proofs; the reader who likes analysis and cares only about $k = \mathbb{C}$ can skip the rest of this section.

Algebraic proof of Proposition 12 in characteristic zero. Let t and u be two uniformizers at x. Let

$$\omega = \frac{a_{-N}du}{u^N} + \frac{a_{-N+1}du}{u^{N-1}} + \cdots + \frac{a_{-1}du}{u} + \eta$$

where η is regular at x. We must show that the residue of ω with respect to t is a_{-1}. Since residue is linear, it suffices to carry out three computations:

Computation 1: For $m \geq 2$, the t-residue of $\frac{du}{u^m}$ is 0. This follows from the fact that $\frac{du}{u^m} = -\frac{1}{m-1}d\frac{1}{u^{m-1}}$ and the easy fact that the residue of df is always 0. Note that this computation uses the fact that k has characteristic 0, in order to divide by $m-1$.

Computation 2: If η is regular at x, then the t-residue of η is 0. This is obvious.

Computation 3: For u a uniformizer, the t-residue of $\frac{du}{u}$ is 1. Let $u = ft$ with f a unit. Then $\frac{du}{u} = \frac{dt}{t} + \frac{df}{f}$. The t-residue of $\frac{dt}{t}$ is 1, and $\frac{df}{f}$ is regular at x. □

Proposition 16. *Proposition 13 holds on* $\mathbb{P}^1$.

Proof. Any rational 1-form on $\mathbb{P}^1$ is of the form $f(t)dt$ for some rational function $f(t)$. Using partial fraction decomposition, we can write $f(t)$ as a sum of monomials in t and functions of the form $1/(t-a)^r$. It is easy to check that $t^m dt$ has all residues equal to 0 for $m \geq 0$, that $dt/(t-a)^r$ has all residues equal to 0 for $r \geq 2$ and that $dt/(t-a)$ has precisely two poles (at a and ∞) with residues 1 and -1. □

We now aim to reduce the general case to that of $\mathbb{P}^1$. Choose a separable noether normalization $\pi : X \to \mathbb{P}^1$. Let $\mathrm{Tr}(\pi)$ to denote the standard trace map from $\mathrm{Frac}(X)$ to $\mathrm{Frac}(\mathbb{P}^1)$. We define trace of 1-forms as follows: Choose a nonzero rational 1-form η on $\mathbb{P}^1$ and write $\omega = f\pi^*\eta$. (Since π is separable, $\pi^*\eta \neq 0$.) Then $\mathrm{Tr}_\pi(\omega) = \mathrm{Tr}_\pi(f)\eta$; we leave it to the reader to check that this is independent of the choice of η.

Remark. Let z be a point of $\mathbb{P}^1$ with $\pi^{-1}(z) = \{x_1, \ldots, x_r\}$, with π ramified of degree e_i at x_i. Then $(\mathrm{Tr}_\pi g)(z) = \sum e_i g(x_i)$, for any $g \in \mathrm{Frac}(X)$ which is regular at $x_1, x_2, \ldots, x_r$. It is difficult to give a similar description of $\mathrm{Tr}_\pi\omega$ in general. If π is unramified at the x_i, then we have $\mathrm{Tr}_\pi(\omega)_x = \sum_i (\pi_i^*)^{-1}\omega_{x_i}$ where π_i^* denote the pull back $T_z^*\mathbb{P}^1 \to T_{x_i}^*X$.

Proposition 17. *For* $\theta \in \mathrm{Frac}(X)$*, we have* $\mathrm{Tr}_\pi d\theta = d\mathrm{Tr}_\pi\theta$.

Proof. The conceptual proof is to use the fiber-wise description of trace in the above remark. We also provide an unenlightening direct proof. Let $\theta^n + a_{n-1}\theta^{n-1} + \cdots + a_1\theta + a_0 = 0$ be the minimal polynomial of θ over $\mathrm{Frac}(\mathbb{P}^1)$. Then

$$d\theta = -\frac{\theta^{n-1}da_{n-1} + \cdots + \theta da_1 + da_0}{n\theta^{n-1} + (n-1)a_{n-1}\theta^{n-1} + \cdots + a_1}.$$

The denominator is nonzero because π is separable. So

$$\mathrm{Tr}_\pi d\theta = -\sum_{j=0}^{n-1} \mathrm{Tr}_\pi \left(\frac{\theta^j}{n\theta^{n-1} + (n-1)a_{n-1}\theta^{n-1} + \cdots + a_1} \right) da_j.$$

We have $a_1 = -\mathrm{Tr}_\pi \theta$, so we are done if we show

$$\mathrm{Tr}_\pi \left(\frac{\theta^j}{n\theta^{n-1} + (n-1)a_{n-1}\theta^{n-1} + \cdots + a_1} \right) = \begin{cases} 1 & j = n-1 \\ 0 & 0 \leq j < n-1 \end{cases}.$$

Pass to a splitting field of $\theta^n + a_{n-1}\theta^{n-1} + \cdots + a_1\theta + a_0$, where θ has Galois conjugates θ_1, θ_2, ..., θ_n. We are being asked to prove the identity

$$\sum_{r=1}^{n} \frac{\theta_r^j}{\prod_{s \neq r}(\theta_r - \theta_s)} = \begin{cases} 1 & j = n-1 \\ 0 & 0 \leq j < n-1 \end{cases}.$$

One slick proof is to note that this is a formal polynomial identity, so it is enough to prove it when θ_1, ..., θ_n are in k. Use Proposition 16, applied to the 1-form $z^r(z-\theta_1)^{-1}(z-\theta_2)^{-1}\cdots(z-\theta_n)^{-1}dz$. ☐

Proposition 18. *For any $z \in \mathbb{P}^1$, we have $\mathrm{res}_z \mathrm{Tr}_\pi(\omega) = \sum_{x \in \pi^{-1}(x)} \mathrm{res}_z(\omega)$.*

Proof of Proposition 18 in characteristic zero. Let π be unramified of order e_i at x_i. Let t be a uniformizer at t. Note that $\pi^*(dt/t)$ has a simple pole at x_i with residue e_i. In particular, the residue at x_i is nonzero, since we are in characteristic zero.

Choose a rational function f on X such that $\omega - df$ has only (at most) simple poles at x_i. Define $g \in \mathrm{Frac}(X)$ by $\omega - df = g\pi^*(dt/t)$. Then g is regular at x_i since $\pi^*(dt/t)$ has nonzero residue at each x_i. So it is enough to prove Proposition 18 for df and for $g\pi^*(dt/t)$.

By Proposition 17, $\mathrm{Tr}_\pi df = d\mathrm{Tr}_\pi f$. Since closed 1-forms have residue 0 everywhere, Proposition 18 for df is simply the identity $0 = \sum_{x \in \pi^{-1}(z)} 0$.

We have $\mathrm{Tr}_\pi(g\pi^*(dt/t)) = \mathrm{Tr}_\pi(g)(dt/t)$ so $\mathrm{res}_z \mathrm{Tr}_\pi(g\pi^*(dt/t)) = (\mathrm{Tr}_\pi g)(z)$. We have $(\mathrm{Tr}_\pi g)(z) = \sum_i e_i g(x_i) = \sum_i \mathrm{res}_{x_i}(g\pi^*(dt/t))$, confirming Proposition 18 for $g\pi^*(dt/t)$. ☐

Proposition 13 now follows immediately: $\sum_{x \in X} \mathrm{res}_x \omega = \sum_{z \in \mathbb{P}^1} \mathrm{res}_z \mathrm{Tr}_\pi(\omega)$, and we have already checked the result on $\mathbb{P}^1$.

Proving Propositions 12 and 18 in characteristic p is surprisingly challenging. Serre writes of these results "it seems one can obtain a truly natural proof only by taking the point of view of Grothendieck's general 'duality theorem'." (*Algebraic Groups and Class Fields*, bibliographic notes at the end of chapter II.) If I get the time, I'll write up some unnatural proofs. Otherwise, I refer you to Serre *ibid*, Chapters II.11-13.

7. Conclusion of the proof

We recall the long exact sequence from Proposition 10. Let D be a divisor on X, let p be a point of X and let $D' = D + p$. Then we have long exact sequences

$$\begin{array}{ccccccccccc} 0 & \to & H^0(\mathcal{O}(D)) & \to & H^0(\mathcal{O}(D')) & \to & k & \to & H^1(\mathcal{O}(D)) & \to & H^1(\mathcal{O}(D')) & \to & 0 \\ 0 & \leftarrow & H^1(\Omega(-D)) & \leftarrow & H^1(\Omega(-D')) & \leftarrow & k & \leftarrow & H^0(\Omega(-D)) & \leftarrow & H^0(\Omega(-D')) & \leftarrow & 0. \end{array}$$

The second sequence is Proposition 10 applied to the divisors $K - D'$ and $K - D = K - D' + p$.

We have bilinear pairings between each pair of vertically aligned vector spaces: The Serre pairings in four positions and the nontrivial pairing $k \times k \to k$ in the middle. If you have been careful enough to realize that the middle vector spaces are actually[4] $T_pX^{\otimes(D(p)+1)}$ and $T_pX^{\otimes(-D(p)-1)}$, then you don't have to wonder how to normalize the middle pairing.

We claim that the vertically aligned horizontal arrows are adjoint to each other. Between the first and second columns, this is straightforward. The claim is that $\int f\omega$ means the same thing whether we are considering f and ω as elements of $\mathcal{O}(D)$ and $\Omega(-D')_{U\cap V}$ or as elements of $\mathcal{O}(D')$ and $\Omega(-D)_{U\cap V}$. The same applies between the fourth and fifth columns.

We now check adjointness between the second and third columns. Choose our cover (U, V) so that $p \in V \setminus U$ and let u be a uniformizer at p. Let $f \in H^0(\mathcal{O}(D'))$ and let a be in k. The bottom map $H^1(\Omega(-D')) \leftarrow k$ is defined as follows: Find $\eta \in \Omega(-D)_V$ whose Laurent expansion at p begins $\left(au^{D(p)} + \cdots\right) du$ and consider η as a class in $H^1(\Omega(-D'))$. The map sends a to η. Now we compute $\int f\eta$. At the points of $V \setminus U$ other than p, the form $f\eta$ has no poles. Writing $f = bu^{-D(p)+1} + \cdots$, the residue of $f\eta$ at p is ab. Meanwhile, the map $H^0(\mathcal{O}(D')) \to k$ sends f to b. So the pairing in the third column is also ab. We can make a similar computation between the third and fourth column.

We can rephrase our adjointness results by saying that we have a commutative diagram:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & H^0(\mathcal{O}(D)) & \longrightarrow & H^0(\mathcal{O}(D')) & \xrightarrow{\alpha} & k & \xrightarrow{\beta} & H^1(\mathcal{O}(D)) & \longrightarrow & H^1(\mathcal{O}(D')) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(\Omega(-D))^\vee & \longrightarrow & H^1(\Omega(-D'))^\vee & \xrightarrow{\alpha'} & k & \xrightarrow{\beta'} & H^0(\Omega(-D))^\vee & \longrightarrow & H^0(\Omega(-D'))^\vee & \longrightarrow & 0
\end{array}$$

Our goal is to prove that the vertical arrows are isomorphisms. Replacing D by $K - D$ and taking duals of everything rotates the diagram 180°, so we may concentrate on the left half.

We cut off the right two columns to form a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\mathcal{O}(D)) & \longrightarrow & H^0(\mathcal{O}(D')) & \longrightarrow & \mathrm{Im}(\alpha) & \longrightarrow & 0 \\
 & & \downarrow \sigma & & \downarrow \sigma' & & \hookdownarrow & & \\
0 & \longrightarrow & H^1(\Omega(-D))^\vee & \longrightarrow & H^1(\Omega(-D'))^\vee & \longrightarrow & \mathrm{Im}(\alpha') & \longrightarrow & 0
\end{array}$$

Note that $\mathrm{Im}(\alpha)$ is quite literally contained in $\mathrm{Im}(\alpha')$.

The snake lemma now implies that $\mathrm{Ker}(\sigma) \cong \mathrm{Ker}(\sigma')$. Using this repeatedly, the kernel of $H^0(\mathcal{O}(D)) \to H^1(\Omega(-D))^\vee$ is independent of D. If $\deg D < 0$ then $H^0(\mathcal{O}(D)) = 0$, so $H^0(\mathcal{O}(D)) \to H^1(\Omega(-D))^\vee$ is injective for all D. The remaining challenge is to prove that it is surjective.

Define $q(D) = \mathrm{CoKer}(H^0(\mathcal{O}(D)) \to H^1(\Omega(-D))^\vee)$; we want to show $q(D)$ is zero. From the snake lemma, $q(D) \to q(D')$ is injective. Also, we already noted that $H^0(\mathcal{O}(D)) \to H^0(\mathcal{O}(D'))$ and $H^1(\Omega(-D))^\vee \to H^1(\Omega(-D'))^\vee$ are injective. Applying these injectivity results repeatedly, we obtain

Proposition 19. *If $D \le E$ then $H^0(\mathcal{O}(D))$, $H^1(\Omega(-D))^\vee$ and $q(D)$ inject into $H^0(\mathcal{O}(E))$, $H^1(\Omega(-E))^\vee$ and $q(E)$ respectively.*

We are now closing in on the end of the proof. By Proposition 19, it is enough to show $q(D) = 0$ when $D \ge 0$. Moreover, we may assume that $X \setminus D$ is affine. (In fact, $X \setminus D$ is affine for any nonempty D. If you don't want to show that, choose a noether normalization $\pi : X \to \mathbb{P}^1$ and replace D by $\pi^{-1}(\pi(D))$; then $X \setminus \pi^{-1}(\pi(D))$ is finite over the affine $\mathbb{P}^1 \setminus \pi(D)$.)

Define

$$\begin{array}{lcl}
H^0(\mathcal{O}(\infty D)) & := & \bigcup_{m=0}^\infty H^0(\mathcal{O}(mD)) \\
p(\infty D) & := & \bigcup_{m=0}^\infty H^1(\Omega(-D))^\vee \\
q(\infty D) & := & \bigcup_{m=0}^\infty q(mD)
\end{array}$$

[4]For a 1-dimensional vector space V, and a positive integer r, the notation $V^{\otimes(-r)}$ means $(V^\vee)^{\otimes r}$.

where the unions[5] make sense because of Proposition 19. Using Approximate Riemann Roch and Proposition 11, we deduce

Proposition 20. *The vector space $q(\infty D)$ is finite dimensional.*

Let $U = X \setminus D$ and let V be some other proper open subset of X such that $X = U \cup V$. We'll compute H^1 and thus q using the cover (U, V). Note that $H^0(\mathcal{O}(\infty D))$ is none other than $\mathcal{O}_U$. Of course, our eventual goal is to show that $p(\infty D) = H^0(\mathcal{O}(\infty D))$ and $q(\infty D) = 0$. Before we can do that, we show:

Proposition 21. *The vector spaces $p(\infty D)$ and $q(\infty D)$ naturally have the structure of $\mathcal{O}_U$-modules.*

Proof. Unpacking the definition, $p(\infty D)$ is the set of linear functionals $\phi : \Omega_{U\cap V} \to k$ which vanish on Ω_U and vanish on $\Omega_V(-mD)$ for some D. For any $f \in U$, multiplication by f takes $\Omega_{U\cap V}$ and Ω_U to themselves. Also, if $f \in \mathcal{O}_U(tD)$, then multiplication by f takes $\Omega_V(-mD)$ to $\Omega_V((t-m)D)$. So the functional $(f * \phi)(\omega) := \phi(f\omega)$ is also in $p(\infty D)$. This gives $p(\infty D)$ the structure of an $\mathcal{O}_U$-module.

Looking at the definition of the map $H^0(\mathcal{O}(D)) \to H^1(\Omega(-D))^\vee$, we see that $H^0(\mathcal{O}(\infty D)) \to p(\infty D)$ is a map of $\mathcal{O}_U$-modules, so the cokernel $q(\infty D)$ inherits the structure of a $\mathcal{O}_U$-module. $\square$

We now combine Propositions 20 and 21 to prove the Riemann-Roch theorem. A finite dimensional $\mathcal{O}_U$ module is torsion. Choose $h \neq 0$ in $\mathcal{O}_U$ so that h acts by 0 on $q(mD)$. Let $D' = D \cup \{h = 0\}$ and $U' = X \setminus D'$. Repeat the whole argument with D' in place of D. The vector space $q(\infty D')$ is, then, a $\mathcal{O}_{U'}$-module, and is thus an $\mathcal{O}_U$-module by restriction to $\mathcal{O}_U \subset \mathcal{O}_{U'}$. We get a map of $\mathcal{O}_U$ modules $q(\infty D) \to q(\infty D')$. But h is a unit in $\mathcal{O}_{U'}$, and h acts by 0 on $q(\infty D)$. So a unit acts by 0 on the image of $q(\infty D)$ in $q(\infty D')$ and we deduce that the map $q(\infty D) \to q(\infty D')$ is zero. But, using Proposition 19, this means that $q(\infty D) = 0$. In particular, $q(D) = 0$.

We have shown that $q(D) = 0$. So the Serre duality map $H^0(\mathcal{O}(D)) \to H^1(\Omega(-D))^\vee$ is an isomorphism. We have proved Serre Duality and, thus, the Riemann Roch theorem. **QED**

[5]It would be tempting, but incorrect, to describe $p(\infty D)$ as $\left(\Omega_{U\cap V}/\bigcap_{m\geq 0}\Omega(-mD)_{U\cap V}\right)^\vee$ or as $\left(\lim_{\infty\leftarrow m} H^1(\Omega(-mD))\right)^\vee$. This does not give the same vector space. One must dualize first, and then take the union/direct limit.