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THE DAILY UPDATE – MATH 631, FALL 2014

September 3 Algebraic geometry is wonderful and everyone should learn it!

September 5 Let k be an algebraically closed field. Let X be a subset of k^n. Then we can consider the set of all polynomials in $k[x_1, \ldots, x_n]$ which vanish on X; we call this $I(X)$. Conversely, let J be a subset of $k[x_1, \ldots, x_n]$. We can consider the set of all points $(u_1, u_2, \ldots, u_n)$ in k^n so that $f(u_1, \ldots, u_n) = 0$ for all $f \in J$; call this set $Z(J)$.

$J(X)$ will always be an ideal, meaning that, if g_1, …, g_r are elements of $J(X)$ and f_1, …, f_r are any polynomials, then $f_1 g_1 + \cdots + f_r g_r$ is in $J(X)$. In fact, it is a radical ideal, meaning that if $f^n \in J(X)$ then $f \in J(X)$. When we prove the Nullstellansatz, we will see that the possible outputs of J are precisely the radical ideals. There isn't a simple description of the possible outputs of V, so we make a definition: A subset X of k^n is called ***Zariski closed*** if it is of the form $Z(J)$ for some J. Zariski closed sets are the set of closed sets of a topology (homework!). It follows from the Nullstellansatz that J and Z give bijections between the sets of radical ideals and of Zariski closed subsets of k^n.

If $X \subset k^n$ is Zariski closed, then we define a ***regular function*** on X to be the restriction to X of a polynomial from $k[x_1, \ldots, x_n]$. The ring of regular functions on X is denoted $\mathcal{O}_X$. A ***regular map*** $X \to Y$ is a map all of whose coordinates are regular functions.

September 8 We went through several statements of the Nullstellensatz:

Let k be an algebraically closed field.

Weak version:

Given $g_1, \ldots, g_N \in k[x_1, \ldots, x_m]$, the following are equivalent:
(1) $g_1, \ldots, g_N$ have no common zero in k^m
(2) $\exists f_1, \ldots, f_N \in k[x_1, \ldots, x_m]$ with $f_1 g_1 + \cdots + f_N g_N = 1$.

Strong version:

Given $g_1, \ldots, g_N, h \in k[x_1, \ldots, x_m]$, the following are equivalent:
(1) whenever $g_1 = \cdots = g_N = 0$, then h is also 0.
(2) $\exists f_1, \ldots, f_N \in k[x_1, \ldots, x_m]$ and $M \in \mathbb{Z}^+$ so that $h^M = f_1 g_1 + \cdots + f_M g_M$.

Weak version for ideals:

For an ideal $J \subseteq k[x_1, \ldots, x_m]$, $Z(J) = \emptyset$ if and only if $J = (1)$.

Strong version for ideals:

For an ideal $J \subseteq k[x_1, \ldots, x_m]$ and $h \in k[x_1, \ldots, x_m]$, $h \in I(Z(J))$ if and only if $\exists h \in \mathbb{Z}^+$ so that $h^M \in J$. That is, $\sqrt{J} = I(Z(J))$.

Weak version for maximal ideals:

All maximal ideals of $k[x_1, \ldots, x_n]$ are of the form $\langle x_1 - a_1, \ldots, x_n - a_n \rangle$ for some $(a_1, .., a_n) \in k^n$.

Corollary: The maximal ideals of $k[x_1, \ldots, x_n]/J$ are $\langle x_1 - a_1, \ldots, x_n - a_n \rangle$ for some $(a_1, .., a_n) \in k^n$.

These mean that we can finally prove that (1) Z and I are mutually inverse bijections between Zariski closed subsets of k^n and radical ideals in $k[x_1, \ldots, x_n]$. (2) Isomorphism classes of algebraic sets are in bijection with isomorphism classes of finitely generated radical

k-algebras and (3) MaxSpec(A), which we preliminarily defined as Hom(A, k), is in fact the set of maximal ideals of A.

September 10 The goal of today was to (carefully) go through Arondo's proof of the contrapositive of the forward direction in the statement of the weak nullstellensatz for ideals from last time, i.e.:
Theorem. If $J \subsetneq k[x_1, \dots, x_n]$ is an ideal for $k = \overline{k}$, then there exists $(a_1, \dots, a_n) \in k^n$ such that $f(a_1, \dots, a_n) = 0$ for all $f \in I$.
The idea is to induce on dimension n: we take our ideal $J \subset k[x_1, \dots, x_n]$, take its intersection $J' := J \cap k[x_1, \dots, x_{n-1}]$, finding a point in $Z(J')$ by the inductive hypothesis, and try to lift this to a point in $Z(J)$. Geometrically, this process corresponds to projection. However, this projection can act badly: if we take $J = \langle xy - 1 \rangle$, then $J' = 0$, hence $0 \in Z(J') \subset k^1$, but there is no point $(0, y) \in Z(J) \subset k^2$. The solution is to apply a linear change of variables before taking the intersection: if $f \in I$ is nonzero, then $f(x_1+\lambda_1, x_2+\lambda_2, \dots, x_{n-1}+\lambda_{n-1}, x_n)$ for suitable choice of λ_i is of the form $x_n^d+f_{d-1}x_n^{d-1}+\cdots+f_1x_n+f_0$, where $f_i \in k[x_1, \dots, x_{n-1}]$. The Theorem is then proved by the following:
Lemma. If J is as above and contains an element f of the form $x_n^d + f_{d-1}x_n^{d-1} + \cdots + f_1x_n + f_0$, where $f_i \in k[x_1, \dots, x_{n-1}]$, then every $(a_1, \dots, a_{n-1}) \in Z(I \cap k[x_1, \dots, x_{n-1}])$ lifts to $(a_1, \dots, a_n) \in Z(J)$.
We actually proved the contrapositive. If there does not exist $(a_1, \dots, a_n) \in Z(J)$, then it is possible to construct another polynomial $g \in J$ such that $g(a_1, \dots, a_{n-1}, x_n) = 1$. Taking the resultant of f and g gives a polynomial $R \in J \cap k[x_1, \dots, x_{n-1}]$ such that $R(a_1, \dots, a_{n-1}) \neq 0$.

September 12

We went over the relation of the connectedness of a Zariski closed set in k^n and the existence of nontrivial idempotents in the coordinate ring; Noetherian rings and Hilbert Basis Theorem.

Definition (idempotent). *An element $e \in A$ is idempotent if $e^2 = e$.*

We proved that $X = X_1 \sqcup X_2$, where X_1, X_2 are Zariski closed $\iff$ the coordinate ring $k[X]$ has nontrivial idempotents(i.e. $e \neq 1$ or 0.)

Corollary. *If $X = X_1 \sqcup X_2$, for $\forall f_0, f_1$ regular functions on X_1 and X_2, there exists f regular function on X such that $f\big|_{X_0} = f_0, f\big|_{X_1} = f_1$.*

We also defined **Noetherian ring**: In a commutative ring , TFAE:
(1) $\forall I \subseteq A$ is finitely generated.
(2) $\forall$ chain of ascending ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$, there exists $r \in \mathbb{N}$ such that $I_r = I_{r+1} = I_{r+2} = \cdots$.

Corollary. *If $X_i \subset k^n$ are Zariski closed and $X_1 \supseteq X_2 \supseteq \cdots \supseteq X_n \supset ...$, then there exists $r \in \mathbb{N}$ such that $X_r = X_{r+1} = X_{r+2} = \cdots$.*

Theorem (Hilbert Basis Theorem). $\forall$ *finitely generated k-algebra is Noetherian.*

Remark. *In particular, if $X \subseteq k^n$ is Zariski closed, then $I(X)$ is finitely generated.*

Corollary. *If $X \subseteq k^n$ is Zariski closed, then X has finitely many connected components.*

September 15

A ***decomposition*** of X is $X = X_1 \bigcup X_2$ where X_1, X_2 are nonempty closed subset of X and $X_1, X_2 \neq X$. X is ***irreducible***, if it has no decomposition.

Theorem: A algebraic set X is irreducible if and only if Ω_X is a domain.

Any algebraic set X can be written as $X = X_1 \bigcup X_2 \bigcup \ldots \bigcup X_n$, where X_i is irreducible. The decomposition is unique if we require $X_i \nsubseteq X_j$ for any $i \neq j$. We define a closed subset Z of X to be an ***irreducible component*** of X if Z is closed, irreducible and there does not exist a closed irreducible Z' in X such that $Z \nsubseteq Z'$. These are the X_i which appear in the unique decomposition described above. The facts follow from the following useful Lemma:

Lemma If $X = X_1 \bigcup X_2 \bigcup \ldots \bigcup X_s$, X_i is irreducible and closed. $Z \subset X$ is closed and irreducible. Then $Z \subset X_i$ for some i.

Finally, we make the translation into commutative algebra: Z is an irreducible component if and only if $I(Z)$ is a minimal prime ideal of Ω_x.

September 17

Today we discussed ***Distinguished Open Sets***, which are also called basic open sets. These denote sets having the form $\mathrm{D}(q) := \{x \in X : q(x) \neq 0\}$ where $q \in \Omega_X$ and X is an algebraic set. We have the following useful result:

Lemma: If $U \subseteq X$ is open and $x \in U$, then there is some $q \in \Omega_X$ such that $x \in \mathrm{D}(q) \subseteq U$.

Next we talked about ***regular functions***. Let X be an algebraic set, $\Omega \subseteq X$ open and $x \in \Omega$ with $f : \Omega \to k$. We say that f is ***regular at x*** if there is an open set $U \subseteq \Omega$ with $x \in U$, and regular functions $g, h : X \to k$, $h|_U \neq 0$ and $f|_U = \dfrac{g}{h}|U$.

For example, $\dfrac{1}{x}$ is regular on $k \setminus 0$, yet it is not regular on k.

We ended the class with an important result. It tells us that the definitions of regular functions as **global functions on** X and as **functions with good local properties** coincide.

Theorem: Suppose X is an algebraic set and $f : X \to k$ is regular at every $x \in X$. That is, for every $x \in X$ there exists $x \in U_x$ open and $g_x, h_x \in \Omega_X$ such that $h_x|_{U_x} \neq 0$, and $f|_{U_x} = \dfrac{g_x}{h_x}|_{U_x}$. Then $f \in \Omega_X$.

September 19

Today we discussed a way of determining if two varieties are "the same". That is, we defined a notion of isomorphism of varieties and looked at a few examples. An ***isomorphism*** $\boldsymbol{X} \cong \boldsymbol{Y}$ of varieties is a homeomorphism on the topological spaces, which takes the ring of regular functions to each other.

A ***morphism*** $X \to Y$ is a continuous map ϕ of topological spaces $\phi : X \to Y$ such that for $V \subseteq Y$ open, if f is regular on V, then $f \circ \varphi$ is regular on $\phi^{-1}(V)$.

Next we discussed some examples. One example that is good to keep in mind is that there is an isomorphism $Z(xy - 1) \cong Z(t)$, where the underlying space of the first set is k^2, and k is the underlying space of the second one. The map $\pi : Z(xy - 1) \to k^*$ given by projection gives the isomorphism.

We also touched on some other interesting/useful mathematics. For instance, we showed that the ***Distinguished Open Sets*** $\mathrm{D}(q)$ are actually isomorphic to affine algebraic varieties even though they themselves are but quasi-affine sets as described. Actually, this is used in the proof of the above example. The last thing we touched on was the following result.

Theorem: The regular functions on D(q) are the functions of the form $\dfrac{f}{q^N}, f \in \Omega_X, N \geq 0$.

September 22 Today's class introduced the idea of projective geometry as a natural extension of plane geometry.

Let V be a k-vector space and $V_{\neq 0} := V - \{0\}$. Then $\mathbb{P}(V)$ is defined to be $V_{\neq 0}/k^*$, where the action of k^* on $V_{\neq 0}$ is given by rescaling of vectors. Defining $\mathbb{P}(V)$ as a set of equivalence classes on the topological space $V_{\neq 0}$, allows us to define a topology on $\mathbb{P}(V)$. This is the quotient topology given by the quotient map $q : V_{\neq 0} \to V_{\neq 0}/k^*$. In particular, a set $X \in \mathbb{P}(V)$ is Z-closed if its pre image $q^{-1}(X)$ is Z-closed in $V_{\neq 0}$. We denote $q^{-1}(X)$ by $CX_{\neq 0}$ and we let $CX := CX_{\neq 0} \cup \{0\}$.

A useful tool in studying projective space is the introduction of ***linear charts***. Let $L \subset V$ be a codimension 1 linear space not passing through 0. The composition of maps $L \hookrightarrow V_{\neq 0} \twoheadrightarrow \mathbb{P}(V)$ is an injective map and $q(L)$ is an open subset of $\mathbb{P}(V)$ since is the complement of a closed set (the vanishing set of a single linear polynomial). We will call $q(L)$ a linear chart in $\mathbb{P}(V)$. One notices that the induced Zariski topology on a linear chart is the Zariski topology on L.

One immediate application of projective geometry is a sleek proof of Desargues' theorem which states that if two triangles are in perspective centrally, then they are in perspective axially.

September 24

The goal of today's class was to set up definitions that allow us to work with projective and quasi-projective varieties.

First we discussed some basic definitions for subsets of projective space and subsets of projective space. Let V be a finite dimensional k vector space. Then we write $V_{\neq 0}$ to denote the set $V \setminus \{0\}$, and $\mathbb{P}(V)$ to be $V_{\setminus 0}/k^*$ the projectivization of V. For $X \subseteq \mathbb{P}(V)$ we will write $CX_{\neq 0}$ for the preimage of X in $V_{\neq 0}$ and CX for $CX_{\neq 0} \cup \{0\}$, the ***cone on X***. If L is a space of codimension 1 in V, not through 0, then the image of L in $\mathbb{P}(V)$ is a ***linear chart***.

Next we talked about the topology on $\mathbb{P}(V)$. The Zariski topology on $\mathbb{P}(V)$ is the quotient topology. That is, Z is closed in $\mathbb{P}(V)$ if and only if its preimage $CZ_{\neq 0}$ is closed in $V_{\neq 0}$ if and only if CZ is closed in V. Similarly, Z is open in $\mathbb{P}(V)$ if and only if $CZ_{\neq 0}$ is open in $V_{\neq 0}$ if and only if $CZ_{\neq 0}$ is open in V.

There are three ways of thinking about the topology.

Theorem. *Given $Z \in \mathbb{P}(V)$, the following are equivalent.*

(1) *Z can be defined as the vanishing set of a set of homogeneous polynomials.*
(2) *CZ is closed in V.*
(3) *For a cover of $\mathbb{P}(V)$ by linear charts $\{L_i\}_{i \in I}$, $Z \cap L_i$ is closed in each L_i.*

In the proof of this, we also proved the lemma that linear charts are open in $\mathbb{P}(V)$.

Similarly, we can consider regular functions in three ways.

Definition. *Let $X \subseteq \mathbb{P}(V)$ be Zariski closed, $\Omega \subseteq X$ Zariski open in X, and $f : \Omega \to k$ a function. If $x \in \Omega$ we say that f* **is regular at** *x if any of the following equivalent conditions hold:*

(1) *There exists* $U \subseteq \Omega$ *open with* $x \in U$ *and* $g, h \in S_d$ *(the homogeneous polynomials of degree* d*) such that* $h|_U \neq 0$ *is nowhere zero on* U *and* $(g/h)|_U = f|_U$.

(2) *The pullback of* f *to the quasi-affine set* $C\Omega_{\neq 0}$ *is regular at some preimage of* x.

(3) *For a linear chart* L *containing* x, $f|_{\Omega \cap L}$ *is regular at* x.

A **regular map** *is a map given by regular functions in these senses.*

Note that if $X \subseteq \mathbb{P}^n$ is Zariski closed and we have $h_0, \ldots, h_n \in S_d$ with no common zero on X, then $X \to \mathbb{P}^n$ given by $x = (x_0, \ldots, x_n) \mapsto (h_1(x), \ldots, h_m(x))$ is regular. However, it is not true that all regular maps can be represented this way. Regularity is a local condition and must be checked on charts.

September 26

We reviewed some notation and gave examples of morphisms of projective varieties.

Example (Change of co-ordinates)**.** Any invertible linear transformation $T : V \to V$ induces a map $\mathbb{P}(V) \to \mathbb{P}(V)$ by $[x_0 : \ldots : x_n] \mapsto [L_0(x) : \ldots : L_n(x)]$ where $x = (x_0, \ldots, x_n)$ and $L_i \in V^*$ is given by $\sum_{j+0}^{N} a_{ij} x_j$.

Example. We can have $f : \mathbb{P}^1 \to \mathbb{P}^2$ by $[s : t] \mapsto [s^2 : st : t^2]$.

This is well-defined:

- $f([s : t]) \neq [0 : 0 : 0]$ because $s \neq 0$ or $t \neq 0$.
- $[s : t] = [\lambda s : \lambda t] \mapsto [\lambda^2 s^2 : \lambda^2 st : \lambda^2 t^2] = [s^2 : st : t^2]$

The image of f is a curve C contained in $Z(xz - y^2) \subset \mathbb{P}^2$, because $s^2 t^2 = (st)^2$. So we have maps

$$\mathbb{P}^1 \to C \to \mathbb{P}^2$$

. In the chart U_s, we have

$$[1 : \frac{t}{s}] \mapsto [1 : \frac{t}{s} : (\frac{t}{s})^2]$$

So we are effectively mapping $\mathbb{A}^1 \to C \to \mathbb{A}^2 = U_x$ mapping by

$$\frac{t}{s} \to (\frac{t}{s}, (\frac{t}{s})^2),$$

in other words $a \mapsto (a, a^2)$. In this chart, $zx - y^2 = z - y^2$ and the map is clearly surjective. An identical argument shows that $U_t \to U_z$ is surjective.

Is f regular? Yes - because it is (obviously) regular on each affine chart!

Example. Let's take the same curve, $C = Z(xz - y^2) \subset \mathbb{P}^2$ and map $g : C \to \mathbb{P}^1$ by $[x : y : z] \mapsto [x : y]$ when $x \neq 0$ and $[x : y : z] \mapsto [y : z]$ when $z \neq 0$.

This is well-defined:

- Take $[x : y : z] \in C$. If $x = z = 0$, then $y = 0$ which is impossible. So either $x \neq 0$ or $z \neq 0$, and hence $[x : y] \in \mathbb{P}^1$ or $[y : z] \in \mathbb{P}^1$.
- What if $x, z \neq 0$ simultaneously? If $[x : y : z] \in C = Z(xz - y^2)$ and $x, z \neq 0$, then $[x : y] = [zx : zy] = [y^2 : zy]$. Since $x, z \neq 0$, we have $y \neq 0$, so $[y^2 : zy] = [y : z]$, as required.

Now, we ask if g is a map of varieties. The image of g on $U_z \subset \mathbb{P}^2$ is given by mapping

$$[\frac{x}{z} : \frac{y}{z} : 1] \mapsto [\frac{y}{z} : 1]$$

and so we are mapping $U_z \cap C \to U_t$. In the affine chart, this is given by

$$(\frac{x}{z}, \frac{y}{z}) \mapsto \frac{y}{z}$$

This is clearly regular, and an identical argument works for U_x. Also, as noted above, $C \subset U_x \cup U_z$ because on C, we must have at least one of $x \neq 0$ or $z \neq 0$. Hence, g is regular.

Notice that f and g above are mutually inverse, so $\mathbb{P}^1$ is isomorphic to the conic $Z(xz - y^2)$ under f. These maps can be generalized, and exactly the same argument shows that the Veronese embeddings of $\mathbb{P}^1$ in $\mathbb{P}^d$ is an isomorphism (of projective varieties) onto its image.

Definition. The d^{th} Vernoese mapping of $\mathbb{P}^1$ is defined by $\mathbb{P}^1 \to \mathbb{P}^d$ by $[s : t] \mapsto [s^d : s^{d-1}t : \ldots : st^{d-1} : t^d]$.

The image of this map above is called the rational normal curve of degree d. More generally, there is a d^{th} Veronese map on *any* projective space $\mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$ defined by $[s_0 : \ldots s : n] \mapsto [\ldots : s_0^{a_0} \cdot \ldots \cdot s_n^{a_n} : \ldots]$ where $\sum a_i = d$ (that is, the image is the list of all monomials of degree d in the coordinates $s_0, \ldots, s_n$). This is also an isomorphism onto its image.

September 29

Today we discussed about a notion of $X \times Y = \{(x, y) : x \in X, y \in Y\}$. We will first define a Zarisko topology in a product space in abstract way, and then we will give an alternative definition using Segre map.

Example. For affine varieties, topology and regular functions are pretty straightforward.

For $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$, where $X = Z(f_1, \cdots f_r)$ and $Y = Z(g_1, \cdots, g_s)$, $X \times Y := Z(f_1, \cdots, f_r, g_1, \cdots, g_s)$.

Note that this is **NOT** a product topology.

Now, we define a Zariski topology in a product space.

Definition. For quasi-projective varieties X and Y, $Z \subset X \times Y = \{(x, y) : x \in X, y \in Y\}$ is **Zariski closed** if for some open covers $X = \bigcup_i U_i$ and $Y = \bigcup_i V_i$, by affines, $Z \cap (U_i \times V_i)$ is closed in $U_i \times V_i$.

Definition. (Regular functions) For an open set $\Omega \subset X \times Y$, a function $f : \Omega \to k$ is **regular** if for some open affine covers $X = \bigcup_i U_i$ and $Y = \bigcup_i V_i$, the restrictions $f|_{(U_i \times V_i) \cap \Omega}$ are regular.

We can give an alternative definition using the Segre map. Here, we want to see $\mathbb{P}^m \times \mathbb{P}^n$ as a projective variety inside $\mathbb{P}^{mn+m+n}$.

Consider a map $\sigma : \mathbb{P}^m \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{mn+m+n}$ such that

$$\sigma : [x_0 : \cdots : x_m] \times [y_0 : \cdots : y_n] \mapsto \begin{pmatrix} x_0y_0 & \cdots & x_0y_n \\ \vdots & \ddots & \vdots \\ x_my_0 & \cdots & x_my_n \end{pmatrix}$$

$$\text{Label } \mathbb{P}^{mn+m+n} \text{ as } \begin{pmatrix} z_{00} & \cdots & z_{0n} \\ \vdots & \ddots & \vdots \\ z_{m0} & \cdots & z_{mn} \end{pmatrix}$$

This is a map from $\mathbb{P}^m \times \mathbb{P}^n$ to $\mathbb{P}^{mn+m+n}$ and it can be easily shown that it is well defined and injective. Using this map, we can give an alternative definition of Zariski closed set

Definition. (Alternative definition) The Zariski Topology on $\mathbb{P}^m \times \mathbb{P}^n$ is the induced topology on $\mathbb{P}^m \times \mathbb{P}^n$: i.e., $Z \subset \mathbb{P}^m \times \mathbb{P}^n$ is closed if $\sigma(Z) \subset \mathbb{P}^{mn+m+n}$ is Zariski closed.

We can show that σ is injective, and the image of $\mathbb{P}^m \times \mathbb{P}^n$ under the Segre map σ is Zariski closed, and the explicit equations are det $\begin{vmatrix} z_{ik} & z_{il} \\ z_{jk} & z_{jl} \end{vmatrix}$ for all $0 \le i, j \le n$ and $0 \le k, l \le m$. Also, the inverse map $\sigma^{-1} : \sigma(\mathbb{P}^m \times \mathbb{P}^n) \to \mathbb{P}^m \times \mathbb{P}^n$ is also regular (with respect to the first definition). So we can give an alternative definition of regular function.

Definition. A set $Z \subset \mathbb{P}^m \times \mathbb{P}^n$ is Zariski closed if $\sigma(Z)$ is Zariski closed in $\mathbb{P}^{mn+m+n}$, and for an open set $\Omega \subset \mathbb{P}^m \times \mathbb{P}^n$, a function $f : \Omega \to k$ is **regular** if f is regular as a function on the quasi-projective space $\sigma(\Omega)$.

October 1

Our next big result is:

Theorem. *Let B be quasi-projective and X be Zariski closed in $B \times \mathbb{P}^n$.*
Then the image of the projection of X onto B is Zariski closed.

Today we talk about why we care.

Recall that in the proof of the Nullstellensatz, we had $X \subset \mathbb{A}^{n-1} \times \mathbb{A}$. The projection of X onto $\mathbb{A}^{n-1}$ might not be closed. In $\mathbb{P}^n$, it won't be the case. The "infinity points" form a "closure".

Example. Resultants: $\mathbb{P}^1 \times \mathbb{A}^{(m+1)+(n+1)}$

$$X = \{[x : y] \times (f_m, \cdots, f_0, g_n, \cdots, g_0)\}$$

such that

$$f_0 x^m + f_1 x^{m-1} + ... + f_m x^m = 0$$

$$g_0 y^n + g_1 y^{n-1} + ... + g_n x^n = 0$$

Theorem implies that there are equations $R_1, ..., R_t$ in the f_i and g_j s.t. $f(x, y)$ and $g(x, y)$ have a common zero in $\mathbb{P}^1$. (Or equivalently, $R_1(f, g) = R_2(f, g) = ... = R_t(f, g) = 0$)

Example. $X \subset \mathbb{P}^n$, $\phi : X \to B$: a regular map. Then $\phi(X)$ is closed. (Consider the graph $\Gamma_\phi = \{(x, b) : \phi(x) = b\} \subset \mathbb{P}^n \times B$. $\phi(X)$ is projection of Γ_ϕ.

Special Case: $X \subset \mathbb{P}^n$, $x \in \mathbb{P}^n \setminus X$. Can project from x to $\mathbb{P}^{n-1}$.

Example. Determinant: $B = \mathbb{A}^{n^2} \times \mathbb{P}^{n-1}$

$$X = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \times \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} : \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = 0 \right\}$$

So there are some polynomials in entries of a_{ij} which vanish when (a_{ij}) has nonzero kernel.

Example. This is a non-example:

$$\begin{array}{ccccccccc} \mathbb{P}^{n-1} & \times & \mathbb{P}^{n-1} & \times & \mathbb{P}^{n-1} & \times & \mathbb{P}^{n-1} & \to & \mathbb{P}^{n^2-1} \\ \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} & & \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} & & \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} & & \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} & & \end{array}$$

$$(\vec{u}, \vec{v}, \vec{w}, \vec{x}) \to \begin{pmatrix} \\ u \\ \\ \end{pmatrix} \begin{pmatrix} \ v \ \end{pmatrix} + \begin{pmatrix} \\ w \\ \\ \end{pmatrix} \begin{pmatrix} \ x \ \end{pmatrix}$$

First of all, it is not a well-defined map(0 in image. and the image doesn't scale well when each input scales).

Also, as a counter-example, closure of a set of rank 2 tensors may contain rank 3 tensors.

We close with some motivation from topology. Let k be a topological space.

Definition: K is universally closed if for all topological spaces B and all closed $X \subseteq K \times B$, projections of X into B is closed in B. (Note that this is the product topology)

Theorem. *Universally closed* $\Leftrightarrow$ *compact.*

See `http://www.cs.bham.ac.uk/~mhe/papers/compactness-submitted.pdf` for the proof.

October 3

The goal of today's lesson was to give a proof of the following theorem:

Theorem. X is a quasi-projective variety and Y is a Zariski Closed subset of $\mathbb{P}^n \times X$. Let π be the projection map $\pi : \mathbb{P}^n \times X \to X$. Then $\pi(Y)$ is closed.

To begin with, we introduce a preliminary result:

Theorem. If X and Y are affine varieties, a regular map $\pi : Y \to X$ is such that Ω_Y is a finite generated Ω_X module. Then $\pi(Y)$ is closed.

In this context, we call π a **finite map**. Let's consider $X = \text{MaxSpec}A$ and $Y = \text{MaxSpec}B$ with the condition that B is finitely generated A-module. We need to show $\pi(Y)$ is closed in X. Take $x \in X \backslash \pi(Y)$, let m_x be the maximal ideal of functions vanishing at x. Since $x \notin \pi(Y)$, we have $\pi^{-1}(x) = \emptyset$. It follows that $\pi^{-1}(x) = Z(m_x B)$ and $\Omega_{\pi^{-1}(x)} = B/\sqrt{m_x B}$, so we have $B/m_x B = (0)$. Now we can apply the Nakayama's Lemma to obtain some $f \in A$, $f \notin m_x$ such that $f^{-1}(B) = (0)$. This gives a distinguished open set $D(f) \subset X$ such that $x \in D(f)$ and $\pi^{-1}(D(f)) = \emptyset$. So $D(f)$ is a open subset of $X \backslash \pi(Y)$, indicating $\pi(Y)$ is a closed set.

Next, we're going to prove our main theorem. It suffices to check on every open cover of X. So without loss of generality, we may assume X is affine and consider $R = \Omega_X$ and $X = \text{MaxSpec}R$. It is clear what we mean by $CY_{\neq 0}$ in $\mathbb{A}^{n+1} \times X$. Choose some Zariski closed CY in $\mathbb{A}^{n+1} \times X$ with $CY_{\neq 0} = CY \cap (\mathbb{A}^{n+1} \times X)$.

The most natural choice (see homework) would be to take $CY = \overline{CY_{\neq 0}}$. Algebraically, this means $\Omega_{CY} = R[x_0, x_1, \cdots, x_n]/I$, where I is $\langle x_0, x_1, \cdots, x_n \rangle$ saturated (i.e. If $x_i f \in I$ for every $i = 0, 1, \ldots, n$, then $f \in I$.) But this argument will work with any choice of CY.

Let $S = R[x_0, x_1, \cdots, x_n]/I$. Let $x \in X \backslash \pi(Y)$ and let m_x be the maximal ideal of functions vanishing at x. Since $Z(m_x I)$ is the preimage of x in CY, we have $Z(m_x I) =$

$\emptyset$ or $(0, 0, \cdots, 0)$. So $\sqrt{m_x I} = (1)$ or $(x_0, \ldots, x_n)$. Thus, when d is sufficiently large, $\langle x_0, x_1, \cdots, x_n\rangle^{d(n+1)} \subset \langle x_0^d, x_1^d, \cdots, x_n^d\rangle \subset m_x I$.

Let $I = \bigoplus_{d=0}^{\infty} I_d$, we have shown that $I_d/m_x I_d \to (R/m_x)[x_0, x_1, \cdots, x_n]_d$ is surjective, i.e. $(S/m_x S)_d = 0$. Consider $R[x_1, x_2, \cdots, x_n]_d$ as a finitely generated R-module, then by Nakayama's Lemma, there exists $f \in R$, $f \notin m_x$ such that $f^{-1}(S_d) = 0$, where $S_d = R[x_1, x_2, \cdots, x_n]_d/I_d$. It follows that $f^{-1}I_d = f^{-1}R[x_0, x_1, \cdots, x_n]_d$. So every $x^e = x_0^{e_0}x_1^{e_1}\cdots x_n^{e_n}$ of degree d can be written as $x^e = \sum r_i s_i$, where $r_i \in f^{-1}R$ and $s_i \in I_d$. Take N large enough, then we can have $f^N x^e \in I_d$ for all x^e. In particular, $\pi^{-1}(D(f)) = \emptyset$, so $D(f)$ is an open set in $X \backslash \pi(Y)$ containing x, hence $\pi(Y)$ is closed.

October 6

First, we clarified the notation in last problem on the homework; in the formula $\mu^*(u)(g_1, g_2) = \sum_1^N v_i(g_1)e_i(g_2)$, u belongs to Ω_G, for which $\{e_i\}$ is a basis.

We remarked that an alternate proof of regular images of projective varieties being closed can be found in Shafarevich, or in alternate (more detailed) form in Cox, Little and O'Shea, *Ideals, Varieties and Algorithms.*

Next, we asked: how many points are in the fiber of a finite map? Recall our setup: $Y = \operatorname{MaxSpec} B$, $X = \operatorname{MaxSpec} A$, and $\pi : Y \to X$ is regular, so we have a map $\pi^* : A \to B$. We say that π is ***finite*** if π^* makes B into a finitely generated A-module. As an example, we considered $Y = \mathbb{A}^1 = \operatorname{MaxSpec} k[v]$ and $X = \mathbb{A}^1 = \operatorname{MaxSpec} k[u]$, where π is given by $v \mapsto u^2$. We also considered the inclusion $Y = \{(x, y) : x^2 + y^2 = 1\} \hookrightarrow X = \mathbb{A}^2$, which is finite since $k[x, y]/(x^2 + y^2 - 1)$ is a cyclic $k[x, y]$-module. We remarked that *any* closed inclusion is finite.

Proposition. *In the notation above, if $\pi : Y \to X$ is finite, then for all $x \in X$, we have $\#\pi^{-1}(x) < \infty$.*

Pf. Let $\mathfrak{m}_x$ be the maximal ideal of regular functions vanishing at x. Let $R = B/\mathfrak{m}_x B$ and $S = B/\sqrt{\mathfrak{m}_x B}$. If a set generates B as an A-module, then it generates R and S as $A/\mathfrak{m}_x \cong k$-modules, i.e., $\dim_k R, \dim_k S \leq N$. We claim that $\#\pi^{-1}(x) = \dim_k S$ (while remarking also that $\dim_k R = \#\pi^{-1}(x)$, counting multiplicites). To prove the claim, we show that if S is any finite-dimensional reduced k-algebra, then $\operatorname{MaxSpec} S$ is finite. If not, then there exist distinct points $p_1, \ldots, p_{1+\dim_k S}$ in $\operatorname{MaxSpec} S$. For $j = 1, \ldots, 1 + \dim_k S$, let $f_j \equiv 1$ on p_j and $f_i \equiv 0$ on the other p_i. We can extend each f_j to a function on S, which gives $1 + \dim_k S$ linearly independent functions on S, a contradiction. It now follows that $\operatorname{MaxSpec} S$ has cardinality $M < \infty$, so

$$S \cong \underbrace{k \times \cdots \times k}_{M \text{ times}},$$

as desired. ■

By "naive fiber size," we will mean $\#\pi^{-1}(x) = \dim_k S$; by "scheme-theoretic fiber size," we will mean $\dim_k R$.

October 8

Today we looked at examples of finite maps. The map of plane curves $Z(uv) \to Z(uv)$ given by $(u, v) \mapsto (u^2, v)$ shows us that to get a well behaved degree notion we should have an irreducible target. Additionally we considered the projection onto the first coordinate $\pi : V(uv, v(v-1)) \to \mathbb{A}^1$. In this situation one fiber is infinite, and so we will be more interested in maps where each irreducible component maps onto a dense subset of the image.

Why is density the right condition? For a general affine projection $\pi : Z_1 \cup \ldots Z_r \to X$ where each Z_i and X is irreducible, we proved that $\Omega_{Z_1 \cup \ldots Z_r}$ is a torsion free Ω_X module under the map π^* exactly when each $\pi(Z_i)$ is dense. One can show this by using the fact that $\pi^*(f)$ is a zero divisor for any nonzero f vanishing on some $\overline{\pi(Z_i)} \subsetneq X$.

Another example we considered is the curve parameterization of $X = Z(y^2 = x^2 + x^3)$ given by $t \mapsto (t^2+1, t(t^2+1))$ under an algebraically closed field of characteristic besides two. The fiber of every point save the origin consists of a single point, the fiber over the origin contains two points. This sort of situation occurs when the coordinate ring of the target is not integrally closed. Note that $\alpha = \frac{y}{x}$ satisfies $\alpha^2 = 1 + x\alpha$. Also note that the degree of this polynomial corresponds to the degree of the fiber over the origin. The integral closure of O_X is the coordinate ring of a curve that projects onto X mapping two of its points to the origin.

The Frobenius map $\mathbb{A}^1 \to \mathbb{A}^1$ taking $t \mapsto t^p$ over a characteristic p field has one point fibers over each point. (To see this note that the Frobenius endomorphism is linear over the prime field and compute the kernel.) On the other hand its scheme theoretic fibers are dimension p.

Yet another example: Let $Y = Z(wy = wz = xy = xz = 0) \subseteq \mathbb{A}^4$ and take $f : Y \to A^2$ given by $(w, x, y, z) \mapsto (w - y, x - z)$. The fibers of this map consist of two points almost everywhere but the origin, where it consists of one. Geometrically, two planes intersecting at a special point are being rotated by f to coincide. However, we can compute the scheme theoretic dimension at the origin and find that $\Omega_Y/(w - y, x - z) \cong k[x, y]/(x^2, xy, y^2)$ is three dimensional.

We define $\deg \pi := \dim_{Frac(A)} B \otimes_A Frac(A)$ for $\pi : \text{MaxSpec}\, B \to \text{MaxSpec}\, A$.

Here are some properties of scheme-theoretic length:

1. It is upper semi-continuous. That is, every point in the domain achieves a local maximum. Stated in a different way $\{x : \text{scheme theoretic length of } \pi^{-1}(x) \geq r\}$ is closed
2. If $\pi : X \to Y$ is finite and Y is irreducible then the scheme theoretic length of any point in the image is at least $\deg \pi$.
3. There is equality on some non-empty open set.

If A is a normal domain, $\pi : \text{MaxSpec}\, B \to \text{MaxSpec}\, A$, B under π^* becomes a torsion-free A-module, and B/A is finite; then we can say of the naive fiber size (compare with the points above):

1. It is lower-semicontinuous.
2. Naive fiber size is at most $\deg \pi$.
3. If the base field has characteristic 0, then on some dense open set in the image where naive fiber size agrees with the degree of the map. (Recall the Frobenius example above).

October 10

The goal of today's class is to prove the 3 results on the scheme-theoretic length and naive fiber size of a finite map, which we had discussed last time.

Theorem. Let $Y = \mathrm{MaxSpec}(B), X = \mathrm{MaxSpec}(A)$ be affine varieties and let $\pi \colon Y \to X$ be a finite map. Let $K = \mathrm{Frac}(A)$ and let $\deg(\pi) = \dim_K(B \otimes_A K)$. Let $\#\pi^{-1}(x)$ denote the scheme-theoretic length.

1. For any $r > 0$, the set $\{x \in X \colon \#\pi^{-1}(x) \geq r\}$ is closed.
2. If X is irreducible, then the scheme length satisfies $\#\pi^{-1}(x) \geq \deg(\pi)$.
3. If X is irreducible, there is a nonempty open $U \subseteq X$ on which the scheme length is equal to $\deg(\pi)$.

Proof of (1): We will show that the complement $\{x\colon \dim B/\mathfrak{m}_x B < r\}$ is open: let $e_1, \dots, e_s$ be a lift of a k-basis of $B/\mathfrak{m}_x B$, then consider the A-module map $A^{\oplus s} \to B$ given by $(a_1, \dots, a_s) \mapsto a_1 e_1 + \ldots + a_s e_s$. By hypothesis, the induced map $k^{\oplus s} \to B/\mathfrak{m}_x B$ is surjective. Nakayama's lemma gives $f \in A \backslash \mathfrak{m}_x$ such that the induced map on the localizations $(f^{-1}A)^{\oplus s} \to f^{-1}B$ is a surjection. Then, for any $x' \in D(f)$, the map

$$(f^{-1}A/\mathfrak{m}_{x'})^{\oplus s} \simeq (A/\mathfrak{m}_{x'})^{\oplus s} \simeq k^{\oplus s} \longrightarrow f^{-1}B/\mathfrak{m}_{x'} f^{-1}B.$$

In particular, $\dim_k(B/\mathfrak{m}_{x'}B) \leq s$ for any $x' \in D(f)$. ☐

Proof of (2): Again let $e_1, \dots, e_s \in B$ be a lift of a k-basis of $B/\mathfrak{m}_x B$, then $e_1, \dots, e_s$ spans $f^{-1}B$ as an $f^{-1}A$-module, for f as above. Therefore, the e_i's span $B \otimes_A K$ as a K-module; in particular, $s \geq \deg(\pi)$. ☐

Proof of (3): Let $e_1, \dots, e_N$ generate B as an A-module, so $e_1, \dots, e_N$ span $B \otimes_A K$ as a K-vector space. Reorder the e_i's so that $e_1, \dots, e_M$ is a K-basis for $B \otimes_A K$ for some $M \leq N$. Thus for $M < j \leq N$, we can write $e_j = \sum_{i=1}^{M} \frac{a_{ij}}{f_{ij}} e_i$ for some $\frac{a_{ij}}{f_{ij}} \in K$, and let $F = \prod_{i, M<j\leq N j} f_{ij}$. Notice that $e_{M+1}, \dots, e_N$, considered as elements of $F^{-1}B$, are in the $F^{-1}A$-span of $e_1, \dots, e_M$. For any $x \in D(F)$, the $e_1, \dots, e_M$ span $B/\mathfrak{m}_x B$, so $\#\pi^{-1}(x) \leq M = \deg(\pi)$. (2) gives the opposite inequality. ☐

Theorem. Let A be an integrally closed domain and let B be finitely-generated and torsion-free as an A-module. Let $\#\pi^{-1}(x)$ now denote the naive fiber size.

1. For any $r > 0$, the set $\{x \in X \colon \#\pi^{-1}(x) \leq r\}$ is closed.
2. $\#\pi^{-1}(x) \leq \deg(\pi)$.

The proof of these results relies on the following commutative algebra lemma. (See `http://www.math.lsa.umich.edu/~speyer/631/GaussLemma.pdf`.)

Lemma. For $\theta \in B$, let $g(t)$ be the monic minimal polynomial of θ over K. Then,

(a) $g(t) \in A[t]$.
(b) The kernel of the map $A[t] \to B$ given by $t \mapsto \theta$ is the ideal $(g(t))$.
(c) Define $\psi \colon Y \to X \times \mathbb{A}^1$ by $x \mapsto (\pi(x), \theta(x))$, then $\mathrm{im}(\psi) = Z(g)$.

See `http://mathoverflow.net/questions/182863/lower-semicontinuity-of-naive-fiber-size` for an MO post discussing better proofs of (1), as well as an outline of the proof from class.

Added by David Speyer: I really thought the proof of (2) was in Shavarevich, but I don't see it. So: Let $\#\pi^{-1}(x) = c$. Choose $\theta \in B$ taking c distinct values on the points of $\pi^{-1}(x)$. Let $g(t)$ be the minimial monic polynomial of θ over K, then $\deg g \leq \deg \pi$. We have $g(\theta) = 0$ everywhere on Y. In particular, the reduction of $g(t)$ modulo $\mathfrak{m}_x$ vanishes at all the values of θ on $\pi^{-1}(x)$. So $\deg g \geq c$ and we conclude $\deg \pi \geq c$. ☐

October 15

Recall: if K/k is any field extension, it can be decomposed as

$$\begin{tikzcd} K \arrow[d, dash] \\ k(t_1, \ldots, t_d) \arrow[d, dash] \\ k \end{tikzcd}$$

where $K/k(t_1, \ldots, t_d)$ is algebraic and $k(t_1, \ldots, t_d)/k$ is purely transcendental. The number d is an invariant of K/k, called the transcendence degree.

Definition. If X is an irreducible affine variety, then $\dim X$ is the transcendence degree of the field extension $\mathrm{Frac}(\Omega_X)/k$. If X is any affine variety, then $\dim(X)$ is the the max dimension of its irreducible components. If X is any quasi-projective variety, then $\dim X = \dim U$ for any dense open affine $U \subset X$.

Theorem. If $\varphi \colon X \to Y$ is a regular map with dense image, then $\dim X \geq \dim Y$.

Corollary. *If Y is irreducible and $X \subset Y$ is nonempty and open, then* $\dim X = \dim Y$.

Corollary. *If $\varphi \colon X \to Y$ is regular, then* $\dim \overline{\varphi(X)} \leq \dim X$.

Lemma. (Noether Normalization Lemma) Given $X \subseteq \mathbb{A}^n$, there is a surjective linear map $\pi \colon \mathbb{A}^n \to \mathbb{A}^d$ such that $\pi \colon X \to \mathbb{A}^d$ is surjective and finite.

Claim: $d = \dim X$.

Proof: If $X = \mathrm{MaxSpec}(A)$ is irreducible, then A is a finite torsion-free $k[t_1, \ldots, t_d]$-module. In particular, $\mathrm{Frac}(A)$ is a finite-degree field extension of $k(t_1, \ldots, t_d)$. If X is not irreducible, write $X = \cup_i Z_i$ where the Z_i's are irreducible, then the maps $Z_i \to \mathbb{A}^d$ are finite and at least one must be surjective. $\square$

Corollary. *If X is closed in Y, then* $\dim X \leq \dim Y$.

Theorem. The poset of irreducible subvarieties of a given X is graded by dimension. More precisely,

(1) If Y, Z are irreducible subvarieties of X with $Y \subsetneq Z$, then $\dim Y < \dim Z$.
(2) If Y, Z are irreducible subvarieties of X with $Y \subset Z$ and $\dim Y < \dim Z - 1$, then there exists an irreducible subvariety W of X with $Y \subsetneq W \subsetneq Z$.

This theorem says that we could have defined the dimension $\dim X$ to be the maximal length of a chain $Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_d \subseteq X$ of irreducible subvarieties of X. Similarly, we can define the Krull dimension of a commutative ring A to be the maximal length of any chain $I_0 \supsetneq I_1 \supsetneq \ldots \supsetneq I_d \supset (0)$ of prime ideals.

October 17 We proved that if X is an irreducible affine variety of dimension d, then any irreducible component of $Z(\theta) \in X$ has dimension $d - 1$ where $\theta \in \mathcal{O}_X$.

Warm up: $X = \mathbb{A}^n, \mathcal{O}_X = k[t_1, ..., t_d], \theta = \varphi_1 \varphi_2 \cdots \varphi_r$, where φ_i are irreducible polynomials, then irreducible components of $Z(\theta)$ are $Z(\varphi_i)$, $Z(\theta) = \bigcup Z(\varphi_i)$, need to check: (1) $Z(\varphi_i)$ is irreducible;(2) $Z(\varphi_i) \subsetneq Z(\varphi_j)$ for $i \neq j$.

First Attempt: Pick a Noether normalization $\pi : X \to \mathbb{A}^d$, we would like $\pi(Z(\theta)) = Z(\text{somthing})$. Let $A = k[t_1, .., t_d], B = \mathcal{O}_X$, $\mathrm{Frac}A = K$, $\mathrm{Frac}B = L$. then we have $g(u) =$

$u^N + g_{N-1}u^{N-1} + \cdots g_0$ be some polynomial that θ satisfies over A. Since $\theta \neq 0$, B is a domain, can assume $g_0 \neq 0$. Then $\pi(Z(\theta)) \subset Z(g_0)$.

There are two problems yet to address:

(1) We don't know $\pi(Z(\theta)) = Z(g_0)$. (This will follow from lemmas about integral extensions in commutative algebra.)

(2) We don't know $\pi(Y_i)$ is some component of $Z(g_0)$ (as opposed to a subset of a component). (We could solve this by invoking Krull's height theorem, to make Y_i be height 1; here we solve it by choosing the normalization (also known as projection) wisely.)

But, we know $\dim(Z(\theta)) \leq d-1$ as $\dim\pi(Z(\theta)) \leq \dim(Z(g_0)) = d-1$.

To address (1): define $\eta : X \to \mathbb{A}^d \times \mathbb{A}^1, \eta(x) := (\pi(x), \theta(x))$, then by Gauss's lemma we know the minimal polynomial $g(u)$ of θ over $K = \mathrm{Frac}(A)$ actually is in $A[u]$ and $\eta(X) = Z(g)$. (See `http://www.math.lsa.umich.edu/~speyer/631/GaussLemma.pdf`.) Thus, $\eta(Z(\theta)) = Z(g, u) = Z(g_0, u) \Rightarrow \pi(Z(\theta)) = Z(g_0)$.

For solving (2): We show that, given

- $X \subset \mathbb{A}^n$ of dimension d
- $Y_2, \ldots, Y_r$ of dimension $\leq d-1$ and
- $p \in \mathbb{A}^n$, not in $Y_2, \ldots, Y_r$

we can find a linear projection normalization $\pi : \mathbb{A}^n \to \mathbb{A}^d$ such that

- $\pi : X \to \mathbb{A}^d$ is a Noether normalization
- $\pi : Y_i \to \mathbb{A}^d$ is finite
- $\pi(p) \notin \pi(Y_2) \cup \cdots \cup \pi(Y_r)$.

We only need to consider $\pi^1 : \mathbb{A}^n \to \mathbb{A}^{n-1}$ as the proof of Noether Normalization; think of $\ker(\pi^1)$ as a point in $\mathbb{P}^{n-1}$. We show that the set of "good" kernels contains nonempty open sets and since $\mathbb{P}^{n-1}$ is irreducible, we know their intersection is nonempty and thus get a projection we want. For the kernel to be "good", we need $\ker(\pi^1) \notin \bar{X} \cap \mathbb{P}^{n-1}$, $\bar{Y_2} \cap \mathbb{P}^{n-1}$, $\ldots$, $\bar{Y} \cap \mathbb{P}^{n-1}$ and the line through p and $\ker(\pi^1)$ in $\mathbb{P}^n$ not intersecting $\cup Y_i$.

Then the proof follows naturally as $Z(\theta) = \cup_{i=1}^r Y_i$ irreducible components and we want to show dim Y_1(or Y_i for any $i = 1, ..., r$)$= d-1$. Choose $p \in Y_1$, $p \notin Y_2, \ldots, Y_r$, choose $\pi : X \to \mathbb{A}^d$ as before so that $\pi(p) \notin \bigcup_2^r Y_i$. Let $g_0 = h_1 \cdots h_s$ so $\bigcup Z(h_j) = Z(g_0) = \bigcup \pi(Y_i)$. So some h_j must vanish at $\pi(p)$, say $h_1(p) = 0$. Now $Z(h_1) = \bigcup_1^r (Z(h_1) \cap \pi(Y_i))$ and $Z(h_1)$ is irreducible so $Z(h_1) = Z(h_1) \cup \pi(Y_i)$ for some i. But $\pi(p) \in Z(h_1)$ and $\pi(p) \notin \pi(Y_i)$ for $i \geq 2$, so we must have $Z(h_1) \subseteq \pi(Y_1)$. So $\dim Y_1 = \dim \pi(Y_1) \geq \dim h_1 = d-1$, and we already know $\dim Y_1 < d$, as desired. (And, in fact, $\pi(Y_1) = Z(h_1)$ since Y_1, and hence $\pi(Y_1)$, is irreducible.)

October 20

We can now tie everything together and show that the poset of ideals is graded by dimension. This means that we can show:

Claim 1 X is affine and irreducible, Y is a closed subvariety with $Y \subsetneqq X$, then $\dim T < \dim X$.

Claim 2 X is affine and irreducible, Z is an irreducible closed subvariety with $\dim T < \dim X - 1$, then $\exists$ irreducible Y, *s.t.* $Z \subsetneqq Y \subsetneqq X$.

We can then deduce:

Corollary/Definition If $f_1, f_2, \ldots, f_r$ are regular functions on irreducible X, then every component of $X \bigcap \{f_1 = 0\} \bigcap \{f_2 = 2\} \cdots \bigcap \{f_r = 0\}$ has dimension greater than $dim X - r$. And if f_i is a nonzero divisor on $X \bigcap \{f_1 = f_2 = \cdots = f_{r-1} = 0\}$ for $i = 1, 2, \ldots, r$, then all components have dimension equal to $\dim X - r$. The sequence $f_1, f_2, \ldots, f_r \in A$ called a ***regular sequence*** if f_i is nonzero divisor in $A/\langle \beta_1, \beta_2, \ldots, \beta_{i-1} \rangle$.

Corollary/Definition If $\dim X = d$, $x \in X$, then there are $f_1, f_2, \ldots, f_d$ such that $\{x\}$ is an irreducible component of $Z(f_1, f_2, \ldots, f_r)$. Such a sequence is called a ***system of parameters*** at x.

Corollary If A, B are closed in $\mathbb{A}^n$ and $A \bigcap B \neq \varnothing$, then $\operatorname{codim} A \bigcap B \leq \operatorname{codim} A + \operatorname{codim} B$.

Corollary If A, B are closed in $\mathbb{P}^n$ and $\dim A + \dim B \geq n$, then $A \bigcap B \neq \varnothing$ and $\operatorname{codim} A \bigcap B \leq \operatorname{codim} A + \operatorname{codim} B$.

October 22 Today's goal was to describe the dimension of fibers of a regular map as a function of the base space. More precisely:

Theorem. *Let $\pi \colon Y \to X$ be a regular map of quasi-projective varieties, where $\dim X = m$ and $\dim Y = n$. Then,*

(1) *Assuming Y is irreducible, $\dim \pi^{-1}(x) \geq n - m$ for all $x \in \pi(Y)$.*

(2) *If π is **dominant**, i.e., $\overline{\pi(Y)} = X$, then there exists a nonempty open subset U of X such that $\pi(Y) \supseteq U$ and $\dim \pi^{-1}(x) = n - m$ for $x \in U$. Note that simply the existence a nonempty open U in $\pi(Y)$ is something we didn't know before now: We only knew that $\pi(Y)$ was Zariksi dense.*

(3) *If π factors as $Y \subseteq X \times \mathbb{P}^N \to X$, where Y is closed in $X \times \mathbb{P}^N$, then the set*

$$\{x \in X : \dim \pi^{-1}(x) \geq k\}$$

is Zariski closed for every k. Note this includes the fact that $\pi(Y)$ is Zariski closed.

We saw the two following interesting examples:

(1) Consider the map $\mathbb{A}^2 \to \mathbb{A}^1$ defined by $(x, y) \mapsto x^2 + y^2$. The fiber over 0 looks like it's just one point! The issue is that there is only one point over $\mathbb{R}$, but over $\mathbb{C}$ we actually get the union of two lines $Z((x + iy)(x - iy))$ as expected.

(3) The statement in Shafarevich (3rd ed., Corollary to Thm. I.1.25) only requires that π is surjective and X and Y are irreducible. This is not true[1]. Here is an example to show that "surjective" certainly isn't enough:

Let $Y = ((\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1) \amalg \{\mathrm{pt}\}$ and $X = \mathbb{A}^1$, where $\pi(x, y) = x$ and $\pi(\mathrm{pt}) = 0$. Then, $\{x \in X : \dim \pi^{-1}(x) \geq 1\} = \mathbb{A}^1 \setminus \{0\}$ is not Zariski closed.

In (1) and (2), we freely restrict to affine subsets, since taking closure preserves dimension.

Proof of (1). Take a system of parameters $f_1, f_2, \ldots, f_m$ at x. The fiber $\pi^{-1}(x)$ is then $Z(\pi^* f_1, \ldots, \pi^* f_m)$, and all components have dimension $\geq n - m$ (via induction on the result from 10/17). $\square$

Proof of (2). Embed $\iota \colon Y \hookrightarrow \mathbb{A}^N$. We can then factor π using the graph of π:

$$\begin{tikzcd}
Y \arrow[r, hook, "(\pi{,}\iota)"] \arrow[dr, "\pi"'] & X \times \mathbb{A}^N \arrow[d, "\mathrm{pr}_1"] \\
& X
\end{tikzcd}$$

We can then use the following lemma from commutative algebra:

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[1] See David Speyer's post of MathOverflow: `http://mathoverflow.net/a/184925/33088`

Lemma ("Relative Noether Normalization"[2]). *If* $\pi\colon Y \to X$ *is dominant, there exists a linear map* $\psi\colon \mathbb{A}^N \to \mathbb{A}^d$ *and a non-empty open subset* $U \subseteq X$ *such that in the diagram*

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, hook] \arrow[d, dashed] & U \times \mathbb{A}^N \arrow[dl, "\mathrm{id} \times \psi"'] \arrow[dd, "\mathrm{pr}_1"] \\
U \times \mathbb{A}^d \arrow[dr, "\mathrm{pr}_1"'] & \\
 & U
\end{tikzcd}$$

the dashed arrow defined by restriction of $\mathrm{id} \times \psi$ *is finite and surjective.*

This subset U is then contained in $\pi(Y)$, and $\dim \pi^{-1}(x) = d$. Using the density of U in X, the density of $\pi^{-1}(U)$ in Y, and the finiteness of the dashed arrow above, $d = n - m$. ☐

Proof of (3). We have the factorization

$$\begin{tikzcd}
Y \arrow[r, hook, "\text{closed}"] \arrow[dr, "\pi"'] & X \times \mathbb{P}^n \arrow[d, "\mathrm{pr}_1"] \\
 & X
\end{tikzcd}$$

We induce on $m = \dim X$. We can assume Y is irreducible since the dimension of $\pi^{-1}(x)$ is defined as the maximum dimension of an irreducible component of $\pi^{-1}(x)$. The image $\pi(Y)$ is closed since $\mathrm{pr}_1\colon X \times \mathbb{P}^n \to X$ is a closed map; by replacing X with $\pi(Y)$ we can assume π is surjective. Also, as X is now the image of an irreducible variety, it is irreducible in turn.

If $k \leq n - m$, we are done by (1). Otherwise, by (2) there exists a nonempty open set U in X such that $\dim \pi^{-1}(x) = n - m < k$ for all $x \in X$. Letting $Z = X \setminus U$, we have $\dim Z < \dim X$ (using that X is irreducible). By the inductive hypothesis on $\pi^{-1}(Z) \to Z$ we are done. ☐

October 24

In today's class, we introduced Grassmannians in three different ways. Let's take $V \cong k^n$ as a k dimensional vector space. We want to construct a variety $G(d, V)$ with points correspond to d-planes in V. In particular, when $d = 1$, the variety $G(1, V)$ becomes $\mathbb{P}(V)$. So Grassmanians provide a natural generalization of projective spaces.

First Way The projective space $\mathbb{P}(V)$ can be interpreted as the quotient $k^* \backslash (V - \{0\})$. Similarly, we can define $G(d, V)$ as the quotient by group action of GL_d, i.e.

$$G(d, V) = GL_d \backslash (d \times n \text{ matrix } - \{\text{matrices whose rank} < d\})$$

.

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[2] See `http://www.math.lsa.umich.edu/~hochster/615W10/supNoeth.pdf` for a proof. If one takes a theorem about spaces Y and modifies it to a theorem about families $\mathcal{Y}$ over B whose fibers look like Y, one usually calls the new theorem a relative version of the old one.

Second Way The idea is based on the analogue of projective space $\mathbb{P}(V)$ being covered by linear charts. Choose a d element subset I of $\{1, 2, \cdots, n\}$. We have the containments:

$$\begin{aligned} & \{\text{Full rank } d \times n \text{ matrices}\} \\ \supset\ & \{d \times n \text{ matrices whose } I\text{-indexed columns are linearly independent}\} \\ \supset\ & \{d \times n \text{ matrices with } \mathrm{Id}_d \text{ in columns indexed by } I\} \end{aligned}$$

Clearly we have:

$$\begin{aligned} \mathrm{GL}_d \backslash \{d \times n & \text{ matrices whose } I \text{ columns are linearly independent}\} \\ & \cong \{d \times n \text{ matrices whose } I \text{ columns are the identity}\} \end{aligned}$$

So $G(d, V)$ is covered by $\binom{n}{d}$ copies of $\mathbb{A}^{d(n-d)}$, which we'll call ***Schubert patches***.

We can also do this in a coordinate free way. Given a decomposition $V = A \oplus B$ with $\dim A = d$, $\dim B = n - d$, then we have an injection:

$$\begin{aligned} \mathrm{Hom}_k(A, B) & \hookrightarrow G(d, V) \\ f \quad & \mapsto \{(a, f(a) \in A \oplus B : a \in A\}. \end{aligned}$$

So, for every decomposition $V \cong A \oplus B$, we get a copy of $\mathrm{Hom}(A, B) \cong \mathbb{A}^{d(n-d)}$ in $G(d, V)$.

Third Way We can define $G(d, V)$ by using exterior algebra as below:

$$G(d, V) = \{\omega \in \mathbb{P}(\bigwedge^d V) : \omega \text{ is of the form } v_1 \wedge v_2 \wedge \cdots \wedge v_d\}.$$

We have the following theorem:

Theorem. $G(d, V)$, as defined above, is a Zariski closed subset of $\mathbb{P}(\bigwedge^d V)$.

This theorem gives rise to an embedding $G(d, V) \hookrightarrow \mathbb{P}(\bigwedge^d V)$, which is called *Plucker embedding*. The equations that define $G(d, V)$ in $\mathbb{P}(\bigwedge^d V)$ are called *Plucker relations*. If we choose a basis for $V \cong k^n$, then there are $\binom{n}{d}$ coordinates on $\bigwedge^d(V)$, which are called *Plucker coordinates*. See also Chapter 14 in *Combinatorial Commutative Algebra* by Ezra Miller, Bernd Sturmfels.

Example. We have three ways to describe $G(2, k^4)$. First we can describe $G(2, k^4)$ as

$$GL_2 \Big\backslash \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} \text{ where } (a_{ij}) \text{ has rank 2.}$$

In the third description, this corresponds to

$$\begin{aligned} (a_{11}e_1 + a_{12}e_2 + a_{13}e_3 + a_{14}e_4) \wedge (a_{21}e_1 + a_{22}e_2 + a_{23}e_3 + a_{24}e_4) = \\ \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} e_1 \wedge e_2 + \cdots + \det \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix} e_3 \wedge e_4 \end{aligned}$$

The reader may enjoy verifying that these 6 two by two determinants obey the relation $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$.

In the second description, $G(2, k^4)$ is covered by open linear charts. As an example, we can take $\begin{pmatrix} 1 & 0 & w & x \\ 0 & 1 & y & z \end{pmatrix} \in G(2, k^4)$, which gives us an embedding $\mathbb{A}^{2\times 2} \hookrightarrow G(2, k^4) \hookrightarrow \mathbb{P}^5$.

$$\begin{pmatrix} 1 & 0 & w & x \\ 0 & 1 & y & z \end{pmatrix} \longmapsto (p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34}) = (1 : y : z : -w : -x : wz - xy)$$

October 27

In today's class we introduced Hilbert polynomials.

Definition. *Let A be $k[x_0, \ldots, x_n]$ and consider M a finitely generated, graded A-module. The* **hilbert function** *of M is*

$$h_M^{func}(t) := \dim_k M_t.$$

Theorem (Hilbert)**.** *There is a polynomial $h_M^{poly}(t)$ such that*

$$h_M^{poly}(t) = h_M^{func}(t) \quad \textit{for } t \gg 0.$$

We call such a polynomial the ***Hilbert polynomial*** of M. For $X \subset \mathbb{P}^n$, we will denote by h_X^{poly} the Hilber polynomial of $A/I(X)$.

Example. We consider $X = \mathbb{P}^1$. Since $k[x, y]_t = Span_k\{x^t, x^{t-1}y, \ldots, y^t\}$, we have that $h_X^{poly}(t) = t + 1$.

Example. In general, for $X = \mathbb{P}^d$ we have that $h_X^{poly}(t) = \binom{t+d}{d} = \frac{(t+d)(t+d-1)\cdots(t+2)(t+1)}{d!}$.

Example. For X a conic in $\mathbb{P}^2$, say $X = Z(f)$, we have that

$$\dim_k(\frac{A}{I(X)})_t = \dim_k \frac{k[x,y,z]_t}{f \cdot k[x,y,z]_{t-2}} = h_{\mathbb{P}^2}^{func}(t) - h_{\mathbb{P}^2}^{func}(t-2) = 2t - 1,$$

for large t.

Example. Let X a degree d curve in $\mathbb{P}^2$, we have that

$$h_X^{poly}(t) = \binom{t+2}{2} - \binom{t+2+-d}{2} = dt + \frac{3d - d^2}{2}.$$

We finished the class discussing the following result.

Theorem. *Let $X \subset \mathbb{P}^n$, say $\dim X = d$. Consider $\pi : X \to \mathbb{P}^d$ be a Noether normalization of degree δ. Then*

$$h_X^{poly}(t) = \delta\frac{t^d}{d!} + (\textit{terms of lower order}).$$

We did not prove this result, but we noticed it was equivalent to a purely algebraic one:

Theorem. *Let M be a finitely generated, graded $k[y_0, \ldots, y_d]$ module, with rank δ over $k[y_0, \ldots, y_d]$. (Meaning that $\dim_{k(y_0,\ldots,y_d)} M \otimes_{k[y_0,\ldots,y_d]} k(y_0, \ldots, y_d) = \delta$.) Then*

$$h_M^{poly}(t) = \delta\frac{t^d}{d!} + (\textit{terms of lower order}).$$

As a final remark, we noticed that the fibers of a projection $\mathbb{P}^n \to \mathbb{P}^d$ are copies of $\mathbb{P}^{n-d}$. So δ is the scheme theoretic fiber size of $X \cap \{$ a ***generic*** $\mathbb{P}^{n-d}\}$. By this statement, we mean that there is an open subset U of $G(n-d+1, k^{n+1})$ over which this map is finite and there is a non-empty $U' \subset U$ over which fibers have scheme theoretic length δ. We talked about how our various semi-continuity theorems allowed us to deduce there would be an open set over which this map was finite with fibers of a fixed scheme theoretic length.

October 29

The topic of today's class was Bezout's Theorem.

Theorem (Bezout, Informally). *Given a degree d and degree e curve in the plane, they intersect at de points.*

There are a few caveats with this statement:

- We need to work over projective space.
- We need to work over an algebraically closed field.
- We have to count with multiplicity
- The curves cannot have common factors.

Theorem (Bezout, more formally). *Let k be an algebraically closed field. Let $f(x, y, z)$ and $g(x, y, z)$ be homogeneous polynomials of $\deg d$ and $\deg e$ respectively, with no common factors. Then the dimension of $(k[x, y, z]/\langle f, g\rangle)_t$ is de for t sufficiently large.*

That is, $\#Z(f, g) = \dim_k \Omega_{f=g=0} \leq de$. Notice that $\dim \mathbb{P}^2 = 2$ and $\dim Z(f) = 1$ so $\dim Z(f, g) = 0$ because g is not a zero divisor on f, so we know that $Z(f, g)$ is a finite set of points in $\mathbb{P}^2$. We can choose coordinates so that these points do not lie on the line $z = 0$. Then $k[x/z, y/z] \to \Omega_Z$ the affine coordinate ring, thought of as the open set $z \neq 0$, with coordinates x/z and y/z, subjects onto the coordinate ring of Z the finite set of points in $Z(f, g)$. Note that $k[x, y, z]_t = \text{Span}(x^t, x^{t-1}y, \dots, z^t)$ and dehomogenizing we can see that this is just the span of $(1, x/z, y/z, \dots, (x/z)^t, \dots, (y/z)^t)$ which for t sufficiently large is everything in Ω_Z.

Thus $(k[x, y, z]/I(Z))_t$ is the Ω_Z span of $(1, x/z, y/z, \dots, (x/z)^t, \dots, (y/z)^t)$, and for $t \gg 0$ this is everything. Note that $I(Z)$ may be bigger than $\langle f, g\rangle$. For example, if we have the circle $x^2 + y^2 = 1$ and the line $y = 1$ then $\langle x^2 + y^2 - 1, y - 1\rangle = \langle x^2, y - 2\rangle$ and the function x is not in $\langle x^2, y - 1\rangle$ but x is zero at this point of intersection.

Added by David Speyer What we want to say is that $x^2 + y^2 = 1$ and $y = 1$ intersect in a non reduced scheme of length 2, but we aren't allowed to say that this term. Once this is allowed, the statement will be that the regular functions on the scheme $\{f = g = 0\}$ have dimension de.

Proof of Bezout. Since $f \neq 0$ and $k[x, y, z]$ is a domain, the map multiplication by f from the $(t-d)$th graded piece to the t graded piece is injective. So $\dim_k(k[x, y, z]/fk[x, y, z])_t = dt +$ (stuff) and since g has no common factors with f, it is a nonzero divisor in $k[x, y, z]/fk[x, y, z]$ and therefore the multiplication by g map from the $(t-e)$th graded piece of $k[x, y, z]/fk[x, y, z]$ to the t graded piece is injective. So $\dim_k(k[x, y, z]/(fk[x, y, z] + gk[x, y, z]))_t = dt +$ (stuff) $-$ $(d(t - e)) +$ (stuff) $= de$.

□

We can use Bezout to prove a lot of classical geometry theorems.

Also, degree assigns an integer to each curve, and is multiplicative. This is the first hint we get at the Chow ring. $A^*(\mathbb{P}^2) = A^0(\mathbb{P}^2) \oplus A^1(\mathbb{P}^2) \oplus A^2(\mathbb{P}^2)$ where we are thinking of $A^k(X)$ as the set of formal sums of codimension k subvarieties of X up to an equivalence relation. For example, $A^0(\mathbb{P}^2) = \mathbb{Z}$ (since $\mathbb{P}$ is the only codimension 0 subvariety), $A^1(\mathbb{P}^2) = \mathbb{Z}$ (where we are thinking of this as degrees of curves) and $A^2(\mathbb{P}^2) = \mathbb{Z}$ (where we are thinking of this as numbers of points). So $A^*(\mathbb{P}^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}[\zeta]/\zeta^3$ where ζ is the equivalence class of a line. In the Chow ring, multiplication is roughly intersection.

What if we work in $\mathbb{P}^n$? Then let $f_1, f_2, \dots, f_n \in k[x_0, \dots, x_n]$ be homogeneous polynomials of degrees $d_1, \dots, d_n$.

Theorem. *If $Z(f_1, \dots, f_n)$ in $\mathbb{P}^n$ is zero dimensional, then* $\dim_k(k[x_0, \dots, x_n]/\langle f_1, \dots, f_n\rangle)_t = d_1 \dots d_n$ *for $t \gg 0$.*

What is easy to show is that if f_i is a non-zero divisor in $k[x_0, \dots, x_n]/\langle f_1, \dots, f_{i-1}\rangle$ then $\dim_k(k[x_0, \dots, x_n]/\langle f_1, \dots, f_n\rangle)_t = d_1 \dots d_i$ for $t \gg 0$.

Theorem (Macaulay, 1916)**.** *If $f_1, \dots, f_i$ are homogeneous polynomials in $k[x_0, \dots, x_n]$ and codim$(Z(f_1, \dots, f_i)) = i$. Then $f_1, \dots, f_i$ is a regular sequence.*

Theorem (Cohen, 1946)**.** *Same holds for $f_1, \dots, f_i \in A$, if A is a regular local ring.*

This leads to an important definition (which will not return this term) a ***Cohen-Macaulay ring*** is a ring in which this holds.

October 31 We discussed the proof that a generic cubic in 4 variables contains 27 lines. We mentioned (but did not prove a similar) that given 4 generic fixed lines in $\mathbb{P}^3$, there are be finitely many lines meeting all of them. Intuitively, meeting a specific line is a codimension-1 condition on a moving line in $\mathbb{P}^3$, so meeting 4 of them should cut us down to a finite number of points in $G(2, 4)$.

Let C_3 be the vector space of homogeneous cubics in w, x, y, z. Let $X \subset \mathbb{P}(C_3) \times G(2, k^4)$ be $\{(\text{cubic}, \mathbb{P}^1) : \text{cubic}|_{\mathbb{P}^1} = 0\}$. We claim that the projection $X \to G(2, k^4)$ has fibers $\mathbb{P}^{(\dim C_3 - 4)-1}$, which comes from the 4 coefficients of a cubic on a line. In more detail, using the GL_4 symmetry, we may assume the line is $L = (* : * : 0 : 0)$. If our cubic is $F(w, x, y, z) = \sum F_{ijkl} w^i x^j y^k z^l$, then $F_L = 0 \iff F_{**00} = 0$. There are 4 terms involving just powers of w and x. This is why the fibers have codimension 4.

So $\dim X = (\dim C_3 - 4) - 1 + \dim(G(2, k^4)) = \dim C_3 - 4 - 1 + 4 = \dim C_3 - 1$. Hence $\pi : X \to \mathbb{P}(C_3)$ is a map of varieties of the same dimension. We know that the image of π is closed. Suppose that $\operatorname{codim} \pi(X)$ in $\mathbb{P}(C_3)$ is c.

By the theorem of dimension of fibers, there is a dense open subset of $\pi(X)$ where fibers have dimension $\dim X - \dim \pi(X)$, and everywhere else the fiber dimension is larger. After counting 27 lines on the Fermat cubic (that is, verifying that the fiber of the Fermat cubic is 0-dimensional), we know there is a dense open subset of $\mathbb{P}(C_3)$ with nonempty 0-dimensional fibers. Since X is projective, we know that $\pi(X)$ is closed. So we have deduced that $\pi(X) = \mathbb{P}(C_3)$. In other words, every cubic contains a line. Moreover, there is a dense open subset of cubics with finitely many lines.

How many? The map $\pi : \pi^{-1}(U) \to U$ has finite fibers and factors as inclusion into $U \times G(2, k^4)$ followed by projection onto U, so it is a finite map. We have found a fiber whose naive size is 27, so we basically know that there is an open set over which the naive size is ≤ 27. (More precisely, we need to check that U is normal and every component of $\pi^{-1}(U)$ dominates U. As to the latter, there is only one component, since X is irreducible.)

Once we check that the scheme theoretic size of the fiber over the Fermat cubic is also 27, we will know that the generic cubic has 27 lines.

November 3

In today's class we talked about tangent spaces.

Let V be a finite dimensional k-vector space, X a Zariski closed subvariety of V, and $x \in X$. We want to define a subspace $T_x X \subseteq V$.

Definition. $T_x X = \{\vec{v} \in V : \frac{d}{dt} f(x + t\vec{v}) = 0$ for all $f \in I(X)\}$.

Actually, we do not have to check that $\frac{d}{dt}f(x+t\vec{v})=0$ for all $f\in I(X)$.

Lemma. If $f_1,\cdots,f_r$ generate the ideal $I(X)$, it is enough to check $f_1,\cdots,f_r$.

With this definition, we can consider a tangent space to $y^2=x^2+x^3$ at $(0,0)$, which is just all of V.

Definition. For $X\subseteq V$, $T_xX\subseteq V$. The cotangent space T_x^* is the dual vectorspace which is a quotient of V^*. That is, $T_x^*V=V^*/\mathrm{Span}_{f\in I(X)}((df)_x)$

Zariski observed that $T_x^*\simeq\mathfrak{m}_x/\mathfrak{m}_x^2$.

Theorem. (Zariski) $T_x^*\simeq\mathfrak{m}_x/\mathfrak{m}_x^2$

Note that this localizes: if $D(q)$ is some distinguished open containing x, then $q^{-1}\mathfrak{m}_x/(q^{-1}\mathfrak{m}_x)^2\simeq\mathfrak{m}_x/\mathfrak{m}_x^2$.

For projective varieties, we can define T_xX in any affine chart and get isomorphic results.

Definition. If X is irreducible, and $x\in X$, we have $\dim T_xX\geq\dim X$ (proof in the book). We call x to be a regular point of X if $\dim T_xX=\dim X$. If X is irreducible, X is regular at x if $\dim T_xX=\max\dim Y_i$ where Y_i are components of X that contain x. Moreover, in this case, there is only one component that contains x.

Note that in this class, "nonsingular", "smooth", "regular" are all the same.

November 5

We want to use T_x^*X to check reducedness. Let $X=MaxSpecA$, $x\in X$, $f_1,...,f_d\in\Omega_X$. $f_1(x)=...=f_d(x)=0$.

If $\{x\}=Z(f_1,...,f_d)$, how do we know if $I(\{x\})=\langle f_1,...,f_d\rangle$?

More generally, how to say $\langle f_1,...,f_d\rangle$ is reduced at x?

Definition. Let I be an ideal of A and x a point of $X=\mathrm{MaxSpec}\,A$. We say that I is reduced at x if $\exists f\in A$, $f(x)\neq0$, for which $f^{-1}I$ is reduced in $f^{-1}A$.

In order for this to be a reasonable definition, we need to know two things:

- If A/I has no nilpotents, then $f^{-1}A/I$ has no nilpotents.
- If we have an open cover $X=\cup U_i$, $U_i=D(f_i)$, and $f_i^{-1}A/f_i^{-1}I$ has no nilpotents, then A/I has no nilpotents.

The first fact checks that being reduced at x is a local condition; the second checks that, if X is reduced everywhere, then X is reduced.

Claim: Let $X=\mathrm{MaxSpec}\,A$ and let x be a point of X. Let $f_1,...,f_d\in A$ with $f_1(x)=...=f_d(x)=0$. Then, $\{x\}$ is a reduced component of $Z(f_1,...,f_d)\Leftrightarrow f_1,...,f_d$ span T_x^*X.

Note that requiring that $f_1,\ldots,f_d$ span $\mathfrak{m}_x/\mathfrak{m}_x^2$ is weaker than asking that $f_1,\ldots,f_d$ generate $\mathfrak{m}_x$ as an A-module. For example, look at the equations $v=u(u-1)=0$. Then v and $u(u-1)$ span the cotangent space at $(0,0)$ because the curves are transverse, but they don't generate $\langle u,v\rangle$ because they also intersect at $(1,0)$.

Proof. This is an immediate consequence of Nakayama's lemma. Since $f_1,\ldots,f_d$ span $T_x^*X=m_x/m_x^2$, there exists a g with $g(x)\neq0$, such that $f_1,\ldots,f_d$ generate $g^{-1}m_x$ as a $g^{-1}A$-module. □

We have two ways to work with functions near x.

Definition. $\Omega_{X,x} = \varinjlim_{U \ni x} \Omega_U$.

On a domain, $\Omega_{X,x} = \{\text{functions regular on some neighborhood of x}\}$. In differential geometry, this is "germs of functions at x".

We also have

Definition. Completion: $\widehat{\Omega_{X,x}} = \underset{\infty \leftarrow t}{lim} \Omega_X / m_x^t$

Theorem. *$\Omega_{X,x}$ injects into $\widehat{\Omega_{X,x}}$. If $f(x) \neq 0$, then f is a unit in $\widehat{\Omega_{X,x}}$.*

X is regular $x \Leftrightarrow \widehat{\Omega_{X,x}} \cong k[[T_x^* X]]$.

We always have a surjection $k[[T_x^* X]] \longrightarrow \widehat{\Omega_{X,x}}$.

November 7

Definition. *Let X be closed in $V \cong \mathbb{A}^n$. We define the* **tangent bundle** *$TX \subset V \times V$ to be*

$$\{(x, \vec{v}) : x \in X, \, \vec{v} \in T_x X\}.$$

We remarked the following:

- TX is Zariski closed (its equations are $g(x) = 0$ and $\sum_{i=1}^n v_i \frac{\partial g}{\partial x_i}(x) = 0$ for all $g \in I(X)$, where $\vec{v} = (v_1, \ldots, v_n)$). It is enough to consider g_i generators for $I(X)$.
- We have a projection $TX \to X$ whose fibers are $T_x X$.

Warning (Added by David Speyer) The above equations may not define the radical ideal of TX. For example, if $X = \{(x_1, x_2) : x_1 x_2 = 0\}$ then the equations described above are $x_1 x_2 = x_1 y_2 + x_2 y_1 = 0$. The element $x_1 y_2$ is not in the ideal they generate, but $(x_1 y_2)^2 = (x_1 y_2)(x_1 y_2 + x_2 y_1) - (y_1 y_2)(x_1 x_2)$ is.

Definition. *A* **vector field** *on X is a regular section v of the projection $TX \to X$.*

Definition. *A* **1-form** *is a regular function $\omega : TX \to k$ such that $\omega|_{T_x X}$ is linear on each fiber of $T_x X$. The collection of 1-forms on X is written Ω_X^1.*

We remarked the following:

- If f is a regular function on X, then df is a 1-form (if $(x_1, \ldots, x_n)$ are coordinates on the first copy of V in $TX \subset V \times V$ and $(v_1, \ldots, v_n)$ are coordinates on the second, then $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} v_i$).
- $d(u + v) = du + dv$, $d(uv) = u\,dv + v\,du$, and $d(z) = 0$ for all $z \in k$. $(\star)$

Theorem. *Let $A = \mathcal{O}_X$. We have $\mathcal{O}_{TX} \cong A[df]_{f \in A}/(\star)$.*

Proof. Let R denote the quotient ring on the right hand side. We claim that if $x_1, \ldots, x_n$ generate A/k, then $x_1, \ldots, x_n, dx_1, \ldots, dx_n$ generate R. To see this, note that the x_i generate A, so if $f = p(x_1, \ldots, x_n)$ for some polynomial p, then $df = \sum_{i=1}^n \frac{\partial p}{\partial x_i} dx_i$. Writing $A = k[x_1, \ldots, x_n]/I$ and $S = k[x_1, \ldots, x_n, v_1, \ldots, v_n]/(g(x), \sum_{i=1}^n \frac{\partial g}{\partial x_i}(x) v_i \, \forall g \in I)$, we clearly have maps $S \to R$ and $R \to S$ (where $A \to A \subset S$). Explicitly, the first map is given on generators by $x_i \mapsto x_i$ and $v_i \mapsto dx_i$. The second map is given by lifting $f \in A$ to $p(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$ and taking $df \mapsto \sum_{i=1}^n \frac{\partial p}{\partial x_i} v_i$. We let the reader check that these maps are inverse (it's fun!). $\square$

Correction (Added by David) The ring $A[df]_{f\in A}/(\star)$ may have nonzero nilpotents (as shown above), so it cannot be the ring of regular functions on anything. The right statement is that $\mathcal{O}_{TX}$ is $A[df]_{f\in A}/(\star)$ modulo the ideal of nilpotents.

Being a vector field or a 1-form is a local notion. More precisely, let X be affine, U_i be an open affine cover of X, and for all $x \in X$ let $v(x) \in T_xX$. If $v|_{U_i}$ is a vector field for all i, then v is a vector field.

Proof. We need to show that $v : X \to TX$ is regular, and we can just check regularity on each U_i. This identifies the part of TX over U with TU (there is a little to check here...). ☐

Finally, we noted that for any regular map $f : X \to Y$ with $f(x) = y$, we get maps $f_* : T_xX \to T_yY$ and $f^* : T_y^* \to T_x^*$, along with corresponding globalizations $f_* : TX \to TY$ and $f^* : \Omega_y^1 \to \Omega_x^1$. The last map is the pushforward map on 1-forms. These are functorial.

November 10

Given a linear map $\pi : V \to W$, we examined open sets built from points $x \in X$ where x satisfies the condition of injectivity on the induced map $\pi : T_xX \to W$.

Lemma. $\{x|\pi : T_xX \to W$ is injective$\}$ is open in X.

Proof. (Sketch) The idea of this proof is to look at the projectivization of the kernel of π at each point $x \in X$. To do so, projectivize TX; ie. just as TX sits in $X \times V$, we have $\mathbb{P}(TX) \subseteq X \times \mathbb{P}(V)$. Examine $\mathbb{P}(TX) \cap (X \times \mathbb{P}(k)) \subseteq X \times \mathbb{P}(V)$. The projection of this to X is "where there is a kernel"; *i.e.* we get a closed set, defined by the condition in the statement of the lemma. ☐

Corollary. $\{x \in X|\dim T_xX = \dim X\}$ *is open (i.e. the set of smooth points is open)*

Proof. Since X is irreducible, we know dim $T_xX \geq d$, for all x. We can re-write the condition of equality as $\exists\pi : V \to W$ such that $\dim W = d$ and $\pi : T_xX \to W$ is injective. Thus, $\{x|\dim T_xX = d\} = \bigcup_\pi \{x|\pi : T_xX \to W$ injective$\}$, where the union ranges over $\pi : V \to W$ surjective, with $\dim W = d$, which gives us the set in the statement of the corollary. ☐

Corollary. $\dim T_xX$ *is upper semi-continuous (i.e.* $\{x \in X|\dim T_xX \geq r\}$ *is closed)*

Lemma. Suppose X is irreducible of dimension d. Let $\pi : V \to W$ be a linear surjection with dim $W = d$, $\pi : X \to W$ dominant, and $\mathrm{Frac}(\mathcal{O}_X)/\mathrm{Frac}(\mathcal{O}_W)$ a separable field extension.

Then $\{x \in X|\pi : T_xX \to W$is injective$\}$ is a non-empty open set in X.

Proof. Choose coordinates (using relative Noether Normalization) $(x_1, \dots, x_d, x_1, \dots, y_{n-d})$ on V such that $\pi(x_1, \dots, x_d, y_1, \dots, y_{n-d}) = (x_1, \dots, x_d)$. Let g_i be the minimal polynomials of each y_i over the field $k(x_1, \dots, x_d)$. By separability, $\frac{\partial g_i}{\partial y_i} \neq 0$ on X. Let $U \subseteq X$ be the open set defined by where the $\frac{\partial g_i}{\partial y_i}$ are non-zero. $g_i(x_1, \dots, x_d, y_i)$ is 0 on X, so $dg_i = 0$ on X.

Thus, $\frac{\partial g_i}{\partial y_i}dy_i + \Sigma\frac{\partial g_i}{\partial x_j}dx_j = 0$ on X. (rearranged, we have $dy_i = -(\frac{\partial g_i}{\partial y_i})^{-1}\Sigma\frac{\partial g_i}{\partial x_j}dx_j$ on U.) This tells us that for $x \in U$, T_x^*X is spanned by the 1-forms $dx_1, \dots, dx_d$. So $\pi : T_xX \to W$ is injective (and thus is in fact an isomorphism). ☐

We give a very typical example of why this lemma requires separability:

Example. Let char $k = p$; take the set $X \subset \mathbb{A}^2$ where $X = \{(x,y)|y = x^p\}$. Define $\pi : X \to Y$ via projection onto the y coordinate; this induces a non-separable field extension of fraction fields of coordinate rings. The line $T_{(x,y)}X$ is horizontal for all $(x,y) \in X$; so the set in the above lemma is not non-empty for this non-separable example.

Corollary. *If X is irreducible, $\{x \in X | X$ is smooth at $x\}$ is a non empty open set.*

The idea of the proof of this corollary is to pick a separable Noether Normalization and apply the above lemma. The existence of such a Normalization is proven in a link on Professor Speyer's website.

Corollary. *If X of dimension d is smooth, then TX is a bundle; i.e. there is an open cover U_i of X for which $TU_i \cong U_i \times \mathbb{A}^d$*

November 12

We began by fulfilling an old promise:

Theorem. Let $\pi : Y \to X$ be a separable finite map between varieties of the same dimension. Then there is a dense open set $U \subset X$ so that the naive and the scheme theoretic length of $\pi^{-1}(x)$ match for $x \in U$.

Proof Sketch. Let $\dim Y = \dim X = d$. Take a separable noether Normalization $\psi : X \to \mathbb{A}^d$. Then there is a dense open $U' \subset X$ on which $T_xX \to T_{\psi(x)}\mathbb{A}^d$ is an isomorphism, and a dense open $Y \subset \pi^{-1}(U)$ on which $T_yY \to T_{\pi(\psi(y))}\mathbb{A}^d$ is an isomorphism. (Since π and ψ are separable, so is $\pi \circ \psi$.) So there $T_yY \to T_{\pi(y)}X$ is an isomorphism for $y \in V$. This gives a dense open in Y, but we want a dense open in X. Let $K = Y \setminus V$. Then $\dim K < \dim Y$, so $\pi(K) \neq X$; take $U = X \setminus K$. ☐

We now prove the Sard-Bertini theorem:

Theorem. Let k have characteristic 0. Let $\pi : Y \to X$ be a dominant map of smooth irreducible varieties. Then there is a nonempty open $U \subset X$ so that $\pi^{-1}(x)$ is smooth for $x \in U$.

Proof sketch. Let $\dim X = d$ and $\dim Y = e$.

We first show that there is a nonempty open U in Y such that $\pi_* T_yY \to T_{\pi(y)}X$ is surjective for $y \in U$ (and, hence, $\pi^{-1}(\pi(y))$ is smooth at y.) Using relative noether normalization, write π as $Y \to X \times \mathbb{A}^{e-d} \to X$. Apply the previous theorem to the first map to get a dense open in Y where $T_yY \to T_{\pi(y)}X \times \mathbb{A}^{e-d}$ is bijective; then the projection onto $T_{\pi(y)}X$ will be surjective.

Let $K = \{y \in Y : \text{rank}(\pi_* : T_yY \to T_\pi(y)X) \leq d\}$. We need to show that $\pi(K)$ is not dense in X. Let K_0 be the smooth points of K, it is enough to show that $\pi(K_0)$ is not dense in X. Suppose to the contrary that $\pi : K_0 \to X$ is dominant. Then, applying the result of the first paragraph to K_0, there is a nonempty open subset of K_0 where $T_yK_0 \to T_{\pi(y)}X$ is surjective. But this map factors as $T_yK_0 \to T_yY \to T_{\pi(y)}X$, and $T_yY \to T_{\pi(y)}X$ is not surjective by the definition of X, a contradiction. ☐

November 14

This class was canceled, but here is something like what I would have said. We pointed out (November 7) that being a vector field or a 1-form is a local notion.

Let X be a quasi-projective variety and let U_i be an open affine cover. For every x in X, let $v(x)$ be an element of TX. We will say that v is a ***vector field*** on X if $v|_{U_i}$ is a vector field for each U_i. This is true for one open affine cover if and only if it is true for all of them.

Similarly, for every $x \in X$, let $\omega(x)$ be an element of T_x^*X. We say that ω is a 1-form on X if $\omega|_{U_i}$ is a 1-form for each U_i. Again, this is true for one open affine cover if and only if it is true for all of them.

Let's see some examples. $\mathbb{P}^1$ can be covered by two copies of $\mathbb{A}^1$. Let x be a coordinate on the first $\mathbb{A}^1$, so a vector field on x is of the form $f(x)\frac{\partial}{\partial x}$ for some $f(x) \in k[x]$. If we switch to the other chart, where the coordinate is x^{-1}, then this vector field becomes $-f(x)x^{-2}\frac{\partial}{\partial x^{-1}}$. This is a regular vector field on the other chart if and only if $-f(x)x^{-2}$ is a polynomial in x^{-1}, i.e., if and only if $\deg f \leq 2$. So the space of global vector fields on $\mathbb{P}^1$ is 3 dimensional; they are all of the form $(a+bx+cx^2)\frac{\partial}{\partial x}$. A similar argument shows that there are no nonzero 1-forms on $\mathbb{P}^1$.

Let's look at an example where there are global 1-forms: The elliptic curve $ZY^2 = X^3 + aX^2Z + bXZ^2 + cZ^3$ in $\mathbb{P}^2$. (In this example, k does not have characteristic 2 or 3.) Let U be the open chart $Z \neq 0$. So coordinates on U are given by $x := X/Z$ and $y := Y/Z$, obeying the relation $y^2 = x^3 + ax^2 + bx + c$. We assume that the cubic $x^3 + ax^2 + bx + c$ has no repeated roots. Set $\omega = \frac{dx}{2y} = \frac{dy}{3x^2+2ax+b}$. The first formula for ω shows that ω is regular where $y \neq 0$, the second formula shows that ω is regular where $3x^2 + 2ax + b \neq 0$. Since we are assuming that $x^3 + ax^2 + bx + c$ and $3x^2 + 2ax + b$ have no common roots, this shows that ω is regular everywhere. Since Ω^1_U is generated as an $\mathcal{O}_U$ module by dx and dy, and we have $dx = (2y)\omega$ and $dy = (3x^2 + 2ax + b)\omega$, this shows that Ω^1_U is generated by ω as a $\mathcal{O}_U$ module (and freely so).

Sidenote: What is an actual global formula for ω? Well, you'd have to know how to express 1 as a $k[x]$ linear combination of $x^3 + ax^2 + bx + c$ and $3x^2 + 2ax + b$. Let's just do a example: $y^2 = x^3 + 1$. Then $(x^3 + 1) - \frac{x}{3}(3x^2) = 1$. So

$$\omega = \left((x^3 + 1) - \frac{x}{3}(3x^2)\right)\omega = \left(y^2 - \frac{x}{3}(3x^2)\right)\omega = \frac{y}{2}dx - \frac{x}{3}dy.$$

So, Ω^1_U is a free $\mathcal{O}_U$ module generated by ω. Does ω extend to the whole elliptic curve? The point of the elliptic curve not in U is at $(0 : 1 : 0)$. Local coordinates are $u := X/Y = x/y$ and $v := Z/Y = 1/y$, obeying the relation $v = u^3 + au^2v + buv^2 + cv^3$, from which we derive that

$$\frac{dv}{3u^2 + 2auv + bv^2} = \frac{du}{au^2 + 2buv + 3cv^2 + 1}.$$

After a surprisingly annoying computation, we obtain that

$$\omega = \frac{dy}{3x^2 + 2ax + b} = \frac{dv^{-1}}{3(u/v)^2 + 2a(u/v) + b} = \frac{-v^{-2}dv}{3(u/v)^2 + 2a(u/v) + b} =$$
$$\frac{-dv}{3u^2 + 2auv + bv^2} = \frac{-du}{au^2 + 2buv + 3cv^2 + 1}$$

and the last formula is manifestly regular at $(0, 0)$. In fact, the space of 1-forms on the whole elliptic curve is 1-dimensional, spanned by X. See your homework for a similar example.

Finally, the rant. The morally right thing to do is to define TX, and the way to do it is clear: Cover X by U_i, and then glue together the TU_i. But we aren't allowed to glue abstract varieties, only to embed them all in a common projective space. I tried to do this and screwed up: This was Problem 7 from Problem Set 9. I did find a way to fix the construction, but it is too horrible to give. (See `http://mathoverflow.net/questions/186396`.) Note also that we haven't even mentioned cotangent bundles in this course. They only make sense if X isn't too singular; let's say for now that X is smooth of dimension d. Then the way we should define them is to cover X by U_i for which TU_i is free. Let the gluing between TU_i and TU_j be given by a function $g_{ij} : U_i \cap U_j \to GL_d$. Then we should be allowed to define the cotangent bundle by just using the dual gluing maps: $(g_{ij}^T)^{-1}$. But, again, we aren't allowed to do abstract gluing, and proving that this gives a quasi-projective variety seems to require some serious theorems. So all of this has to wait for next term, when I am allowed to glue together a variety without finding a way to squeeze it into a projective space.

November 17

Why are curves so nice? Here are 3 reasons.

(1) It is easy to tell whether a rational function on a smooth curve is regular at a point (the same is true on a normal variety of any dimension).

(2) It is easy to desingularize curves (it is easy to normalize in any dimension, and it is possible (by Hironaka) to desingularize in characteristic zero).

(3) Rational maps from smooth curves to projective varieties extend to global maps.

Let's elaborate on (1): let X be a 1-dimensional irreducible variety and let x be a smooth point of X. Write $\mathrm{Frac}(X)$ for the rational functions on X, then $\Omega_{X,x}$ is a dvr (that is, the maximal ideal $\mathfrak{m}_x \subset \Omega_{X,x}$ is principal). A generator u_x of $\mathfrak{m}_x$ is called a ***uniformizer*** at x.

For any $f \in \mathrm{Frac}(X)^\times$, define $\nu_x(f)$ to be the unique integer such that $f = u_x^{\nu_x(f)} a$, where $a \in \Omega_{X,x}$ is some unit. Then, $\nu_x \colon \mathrm{Frac}(X)^\times \to \mathbb{Z}$ defines a valuation, meaning that $\nu_x(fg) = \nu_x(f) + \nu_x(g)$ and $\nu_x(f+g) \geq \min\{\nu_x(f), \nu_x(g)\}$ with equality if $\nu_x(f) \neq \nu_x(g)$. Formally, define $\nu_x(0) = \infty$. Then, $f \in \mathrm{Frac}(X)$ is regular at x iff $\nu_x(f) \geq 0$.

Now, consider (2): let $X = \mathrm{MaxSpec}(A)$ be an irreducible affine variety and let $\tilde{A}$ be the integral closure of A in $\mathrm{Frac}(A)$, then $\tilde{X} = \mathrm{MaxSpec}(\tilde{A})$ is the ***normalization*** of X. It is clear that $\tilde{A}$ is radical, and if A is finitely-generated, then so too is $\tilde{A}$ (proof in Shafarevich).

Example: Let $A = k[x,y]/(y^2 - x^3)$, then $\theta := \frac{y}{x}$ satisfies $\theta^2 = x$, so $\tilde{A} = k[\theta]$ is the integral closure of A, where $x = \theta^2$ and $y = \theta^3$.

A variety X is ***normal at*** x if there is an open affine $U = \mathrm{MaxSpec}(A)$ around x, where A is integrally closed in $\mathrm{Frac}(A)$. To check that this definition is local, one should check that: (i) A is integrally closed in $\mathrm{Frac}(A)$, then $f^{-1}A$ is integrally closed in $\mathrm{Frac}(f^{-1}A) = \mathrm{Frac}(A)$; (ii) if $X = \cup_i U_i$ is an open affine cover with $U_i = \mathrm{MaxSpec}(A_i)$ and $X = \mathrm{MaxSpec}(A)$ where each A_i is integrally closed in $\mathrm{Frac}(A_i) = \mathrm{Frac}(U_i)$, then A is integrally closed in $\mathrm{Frac}(A) = \mathrm{Frac}(X)$. Using this, we can talk about normality on quasi-projective varieties using the local definition.

Finally, consider (3): if X is a smooth curve, $U \subset X$ is a dense open subset, and $\varphi \colon U \to Y \subset \mathbb{P}^n$ is a rational map with Y closed in $\mathbb{P}^n$, then φ extends to a regular map on X.

Proof. Let $x \in X \backslash U$, then it suffices to extend φ as a map to $\mathbb{P}^n$ defined near x. Let $\varphi(t) = (\varphi_0(t) \colon \ldots : \varphi_n(t)$ with $\varphi_i \in \mathrm{Frac}(X)$ not all zero, then WLOG say $\nu_x(\varphi_0) \leq \nu_x(\varphi_i)$ for $i = 1, \ldots, n$. We can extend by $(1 \colon \frac{\varphi_1}{\varphi_0} \colon \ldots : \frac{\varphi_n}{\varphi_0})$, which is regular near x. □

In particular, if X, Y are smooth projective curves over k such that $\mathrm{Frac}(X)/k \simeq \mathrm{Frac}(Y)/k$, then $X \simeq Y$.

November 19

Today we discussed Ramification and the Riemann-Hurwitz theorem.

Definition. *Suppose that* X *and* Y *are smooth curves with* $x \in X, y \in Y$ *and* $\pi : Y \to X$ *a non-constant surjective map. Moreover, assume that* u_y *and* u_x *are uniformizers in the local rings* $\Omega_{X,x}$ *and* $\Omega_{Y,y}$ *respectively. We write* $m_x = u_x \Omega_{X,x}$. *If* v_x *is the evaluation map, note that* $v_x(u_x) = 1$.

(1) *We define the* **ramification** *of* π *at* y *to be* $v_y(\pi^* u_x)$.

(2) *Equivalently, we can define the ramification by* $\dim_k(\Omega_{Y,y}/\Omega_{Y,y}m_x)$.

(3) *In the special case* $v_y(\pi^* U_x) = 1$ *we say that* π *is* **unramified** *at* y. *This special case is equivalent to* $d\pi_x : T_y Y \longrightarrow T_x X$ *being an isomorphism. This is in turn equivalent to the dual map* $(d\pi_x)^* : T_x^* X \longrightarrow T_y^* Y$ *being an isomorphism.*

If $v_y(\pi^* u_x) = e$, then $\{1, u_y, u_y^2, ..., u_y^{e-1}\}$ is a basis for the quotient $\Omega_{Y,y}/\Omega_{Y,y}m_x$.

Next we proved the following result. Note that we are still assuming X and Y to be smooth curves:

Theorem. Let π be a finite map of degree n. Then for any $x \in X$ the following relation holds:

$$\sum_{y \in \pi^{-1}(x)} e_y = n.$$

In other words, the scheme theoretic length of $\pi^{-1}(x)$ is always n.

Corollary. *Suppose that* X *is a smooth projective curve and that* $f \in Frac(X)$ *is non-constant. Then* $\sum_{x \in X} v_x(f) = 0$.

Proof. Think of f as a map $X \longrightarrow \mathbb{P}^1$. Then:

$$\sum_{x \in X} v_x(f) = \sum_{y \in f^{-1}(0)} e_y - \sum_{y \in f^{-1}(\infty)} e_y = \deg f - \deg f = 0.$$

□

Let $\omega = f du_x$ be a rational 1-form on X for $x \in X, f \in Frac(X)$ and u_x a uniformizer. We extend the valuation map v_x to rational 1-forms by defining $v_x(\omega) = v_x(f)$. One does need to check that this is well-defined. That is, one needs to check that this extension is independent of the choice of uniformizer u_x. This leads us to the **Riemann-Hurwitz formula:**

Theorem. There is a number g (dependent on X) such that:

$$\sum_{x \in X} v_x(\omega) = (2g-2) \text{ for any non-zero rational 1-form } \omega \text{ on } X.$$

Moreover, in $Char(k) = 0$, if $\pi : Y \longrightarrow X$ is a non-constant map of smooth projective curves, then:

$$2g_Y - 2 = (\deg \pi)(2g_x - 2) + \sum_{y \in Y}(e_y - 1).$$

We call this number g the **genus** of X. We end this discussion with the following example:

Example. We will calculate the genus of $\mathbb{P}^1$. Notice that $\mathbb{P}^1 = U_{y \neq 0} \cup U_{x \neq 0} = \mathbb{A}^1 \cup \mathbb{A}^1$. Also, $\frac{dx}{x}$ is a non-zero rational 1-form on $\mathbb{P}^1$. Furthermore, $y = x^{-1}$. So, in terms of y, $\frac{dx}{x} = \frac{dy^{-1}}{y^{-1}} = -\frac{y^{-2}dy}{y^{-1}} = -\frac{dy}{y}$. Therefore $\frac{dx}{x}$ has a simple pole at $x = 0$, and $-\frac{dy}{y}$ has a simple pole at $y = 0$ $(x = \infty)$. It follows that $\sum_{x \in X} v_x(\omega) = (-1) + (-1) = -2 = 2g - 2 \implies g = 0$.

November 21

Today we proved the Riemann-Hurwitz theorem and introduced some new vocabulary. In all what follows X is a smooth projective curve, ω is a rational 1-form on X and u is an uniformizer at $x \in X$.

Theorem (Riemann-Hurwirtz). *Let k have characteristic 0 and let X and Y be smooth irreducible projective curves. Consider $\pi : Y \to X$ a non-constant map. Then*

$$2g_Y - 2 = \deg(\pi)(2g_X - 2) + \sum_{y \in Y}(e_y - 1).$$

To prove the theorem, one needs to compute $\sum_{y\in Y} v_y(\pi^*\omega)$, for ω a rational 1-form on X such that $\pi^*\omega$ generates T_y^*Y. Then if $\omega = g du_x$, we have that

$$v_y(\pi^*\omega) = v_y(\pi^*g) + v_y(\pi^*du_x) = e_y v_x(\omega) + e_y - 1.$$

Note that we used characteristic 0 to see that π^*u_x vanishes to order $e_y - 1$ at y: If $e_y \equiv 0 \bmod p$, then π^*u_x would vanish to higher order.

Summing over y,

$$\sum_{y\in Y} v_y(\pi^*\omega) = \sum_{x\in X}\sum_{y\in\pi^{-1}(x)} e_y v_x(\omega) + \sum_{y\in Y}(e_y - 1) = \deg(\pi)\sum_{x\in X} v_x(\omega) + \sum_{y\in Y}(e_y - 1)$$

so

$$2g_y - 2 = \deg(\pi)(2g_x - 2) + \sum_{y\in Y}(e_y - 1).$$

Corollary. *For a map $\pi : Y \to X$ as in the theorem, we have that $g_Y \geq g_X$.*

We concluded with a list of definitions:

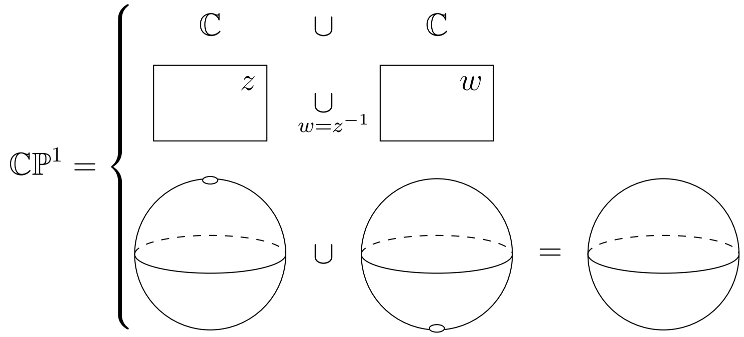
Definition.
1. *A* **divisor** *D is a finite formal $\mathbb{Z}$-sum of points on X*
2. *The* **degree** *of a divisor $D = \sum D(x)x$ is $\deg(D) = \sum D(x)$.*
3. *For $f \in \mathrm{Frac}(X)$, we have that $div(f) = \sum_{x\in X} v_x(f)x$.*
4. *For ω a rational 1-form, we have that $div(\omega) = \sum_{x\in X} v_x(\omega)x$.*
5. *A divisor of the form $div(f)$ is called* **principal**.
6. *A divisor of the form $div(\omega)$ is called* **canonical**.
7. *The divisors D and E are called* **rationally equivalent** *if $D - E$ is principal.*

November 24

Our aim today is to talk about the complex analytic and topological side of algebraic geometry, in particular with respect to genus and the Riemann-Hurwitz theorem.

We start with the following example:

Example. Consider $\mathbb{CP}^1$. Recall we can obtain $\mathbb{CP}^1$ by gluing together two copies of the complex plane $\mathbb{C}$. We can visualize this in the following ways:



We call $\mathbb{CP}^1$ the ***Riemann sphere***.

This example suggests a relationship between algebraic and topological objects:

$$\begin{array}{ccc} \underline{\text{Algebraic objects}} & & \underline{\text{Topological objects}} \\ \left\{\begin{matrix}\text{Smooth curves}\\ \text{over } \mathbb{C}\end{matrix}\right\} & \Longrightarrow & \left\{\begin{matrix}\text{Smooth oriented surfaces}\\ \text{(i.e. dim } 2\ \mathcal{C}^\infty\text{-manifolds)}\end{matrix}\right\} \\ \left\{\begin{matrix}\text{Zariski closed}\\ \text{subsets of } \mathbb{CP}^n\end{matrix}\right\} & \underset{\text{Thm.}}{\overset{\Longrightarrow}{\dashleftarrow}} & \left\{\begin{matrix}\text{Analytically closed}\\ \text{subsets of } \mathbb{CP}^n\end{matrix}\right\} \end{array}$$

We conclude from the implications above that smooth projective curves are smooth orientable compact surfaces. In addition, we have the following non-obvious facts that allow us to fill in the dashed implication in the second line above:

Theorem. (1) *If an algebraic variety over $\mathbb{C}$ is connected in Zariski topology, then it is connected in the analytic topology.*[3]

(2) *If X is a smooth connected compact complex manifold of real dimension 2, then X can be embedded into $\mathbb{CP}^n$,*[4] *and is Zariski closed in $\mathbb{CP}^n$*[5].

Next, we want to understand what elliptic curves correspond to topologically. To do so, we need to understand what ramification corresponds to analytically.

Consider the following regular map of complex algebraic curves, visualized topologically as a map between oriented surfaces:

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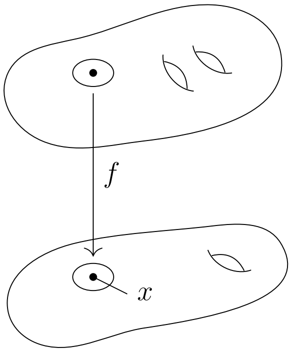
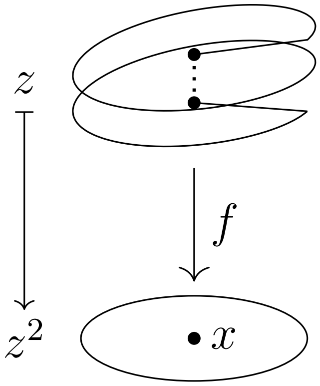
[3] Prof. Speyer said this result is in Serre's "Géométrie algébrique et géométrie analytique"; I couldn't find it there, but it *is* Prop. 2.4 in SGA1, Exp. XII. Added by Prof. Speyer: It certainly follows from GAGA: For example, Theoreme 1 states in particular that $H^0(X, \Omega) \cong H^0(X^{an}, \Omega^{an})$. The left hand side is locally constant functions in the Zariski topology; the right hand side is locally constant functions in the analytic topology. For non-singular X, Proposition 12 states that the singular cohomology group $H^0(X^{an}, \mathbb{C})$ can be computed algebraically, so the number of connected components of X can be as well. But I agree that I don't see this corollary singled out anywhere in GAGA.

[4] See, for example, Thm. 17.22 in Forster's *Lectures on Riemann surfaces.*

[5] This is Chow's Theorem, which states that if $X \subset \mathbb{P}^n$ is a closed $\mathbb{C}$-analytic submanifold, then X is Zariski closed in $\mathbb{CP}^n$. This is Prop. 13, n° 19 in Serre's "Géométrie algébrique et géométrie analytique".



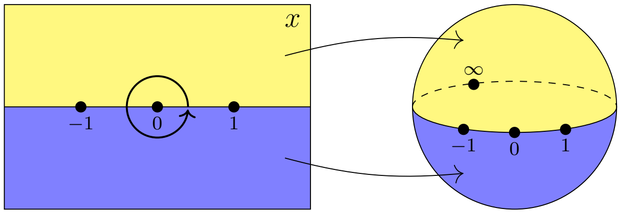
If f is unramified at x, then f induces a local homeomorphism in every sufficiently small analytic neighborhood around x. If f is ramified at x, then f induces the map $z \mapsto z^e$ in local coordinates in every sufficiently small analytic neighborhood around x, where e is the ramification of f at x. Zooming into a neighborhood, the map corresponds to that obtained by making a "branch cut". We visualize this as below for $e = 2$:



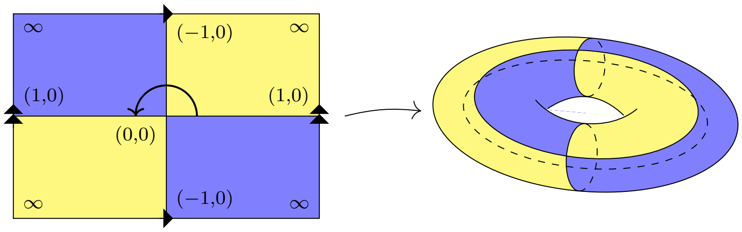
Note that an angle θ upstairs maps to an angle $e \cdot \theta$ downstairs, if we think of going around the neighborhood upstairs as having an angle 2π.

We are now ready to look at elliptic curves.

Example. Consider the elliptic curve $E = \{y^2 = x^3 - x\} \subset \mathbb{C}^2$, together with the projection $p\colon E \to \mathbb{A}^1$ mapping $(x, y) \mapsto x$, which is a degree 2 cover of $\mathbb{A}^1$. This map is ramified at $x = -1, 0, 1$; the map p also extends to a map $\overline{E} \to \mathbb{CP}^1$ where $\overline{E}$ is the projective closure of E in $\mathbb{CP}^2$, and this extension gains one more ramification point at $x = \infty$. As in the diagrams above, we think of $\mathbb{A}^1$ and its compactification $\mathbb{CP}^1$ topologically, marking the ramification points $x = -1, 0, 1, \infty$:



We now want to determine how $\overline{E}$ covers $\mathbb{CP}^1$. To do so, we look at the preimage of the angle drawn around $x = 0$. Above, we recalled that an angle θ maps to an angle $2 \cdot \theta$ in the base space since the ramification is 2 at $x = 0$, so the preimage of this angle is an angle π around $(0, 0)$. By the same argument around the points $-1, 1, \infty$ which also have ramification 2, we obtain the following picture of E and $\overline{E}$:



where each maize region maps to the region in $\mathbb{CP}^1$ above, and similarly for the blue regions. Note that following paths in $\mathbb{CP}^1$ above in the covering space E, we get the identifications of edges labeled above, hence topologically $\overline{E}$ is a torus $S^1 \times S^1$, i.e., it has genus 1.

In general, if $H = \{a_{2h}x^{2h} + a_{2h-1}x^{2h-1} + \cdots + a_0\} \subset \mathbb{C}^2$ is a hyperelliptic curve, then its closure in $\mathbb{CP}^{h+1}$ is a genus $h-1$ surface. Since this is more difficult to visualize, we instead opt to compute the Euler characteristic.

Let $\pi\colon X \to \mathbb{CP}^1$ be a finite cover of degree d; π is ramified over $z_1, \ldots, z_r \in \mathbb{CP}^1$ with ramification $e_1, \ldots, e_r$. Choose a regular CW decomposition of $\mathbb{CP}^1$ with vertexes $z_1, \ldots, z_r$, with V vertexes, E edges, and F faces; note $V - E + F = 2$. The preimage of this CW structure gives a CW structure on X, with dF faces and dE edges as expected, but only $dV - \sum(e_i - 1)$ vertexes because of ramification. We therefore have

$$\begin{aligned} 2 - 2g_X^{\mathsf{top}} = \chi_{\mathsf{top}}(X) &= dF - dE + \Bigl(dV - \sum(e_i - 1)\Bigr) \\ &= d(F - E + V) - \sum(e_i - 1) \\ &= 2d - \sum(e_i - 1) \end{aligned}$$

This is the Riemann-Hurwitz formula, for genus defined as the degree of a rational 1-form!

Conclusion. *Genus, as defined by* $2g - 2 = \deg \operatorname{div}(\omega)$*, is topological genus.*

Corollary. *Over* $\mathbb{C}$*, genus is a non-negative integer.*

We remark that Weil's calculations of degrees of rational 1-forms on curves over fields of positive characteristic prompted him to look for a generalization of genus to algebraically closed fields other than $\mathbb{C}$.

In general, however, working over fields of positive characteristic can get quite cumbersome. We recall from the last problem set that the residue of a rational 1-form is well-defined over a field of characteristic 0. Indeed, this can be seen as an analogue of the residue formula from complex analysis: if γ is a closed loop around $z = 0$,

$$\oint_\gamma f(z)\; dz = (2\pi i) \operatorname{res}_0 f,$$

and if we write $f(z) = \sum a_i z^i$ as a Laurent series, we have $\operatorname{res}_0 f = a_{-1}$. In the algebraic setting, if X is a smooth curve, with u_x a uniformizer at $x \in X$ and ω a rational 1-form, we can write $\omega = \left(\sum a_i u_x^i\right) du_x$, and defined $\operatorname{res}_x \omega = a_{-1}$. While we were able to show this without too much pain in the last problem set for characteristic 0, there is no good proof of this in positive characteristic.[6]

November 26

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[6] Serre in *Groupes algébriques et corps de classes* remarks that the proofs for arbitrary characteristic are "artificielles" (p. 35).

Let X be a smooth connected projective curve. The only regular functions on X are constants. The rational functions on X, called $\mathrm{Frac}(X)$, are a transcendental degree 1 extension of k.

For a divisor D on X, we define

$$H^0(\mathcal{O}(D)) = \{f \in \mathrm{Frac}(X) : \mathrm{div}(f) + D \geq 0\}.$$

The condition is equivalent to saying that for all $x \in X$, we have $\nu_x(f) \geq -D(x)$, where $\nu_x(f)$ is the valuation (number of zeros). We also define

$$H^0(\Omega(D)) = \{\omega \in \mathrm{Frac}(\Omega(X)) : \mathrm{div}(\omega) + D \geq 0\}.$$

As an example, we looked at

$$H^0(\mathcal{O}(2p+q)) = \left\{ f : \begin{array}{c} f \text{ is regular at } x \neq p, q, \text{ has at} \\ \text{most a double pole at } p, \text{ and at} \\ \text{most a single pole at } q \end{array} \right\}$$

Notice that if $\deg D < 0$, then $H^0(\mathcal{O}(D)) = 0$. This is because if $f \in H^0(\mathcal{O}(D)) \setminus \{0\}$, then $\mathrm{div}(f) + D \geq 0$, so $\deg \mathrm{div}(f) + \deg D \geq 0$, giving $\deg D \geq 0$.

Proposition. $\dim H^0(\mathcal{O}(D)) \leq \max\{\deg D + 1, 0\}$

Proof. By induction on $\deg D$. The case $\deg D < 0$ we already noticed. In general, suppose we've proven it for $\deg D = d$. Let $\deg D' = d + 1$, so $D' = D + p$. Then $H^0(\mathcal{O}(D)) \leq H^0(\mathcal{O}(D'))$ and $\dim H^0(\mathcal{O}(D'))/H^0(\mathcal{O}(D)) = 0$ or 1. Now $\dim H^0(\mathcal{O}(D')) \leq \dim H^0(\mathcal{O}(D)) + 1 \leq d + 1 + 1$. ☐

Theorem (Riemann-Roch, often-good-enough version)**.** $\dim H^0(\mathcal{O}(D)) \geq \deg D + 1 - g$.

Theorem (Riemann-Roch)**.** $\dim H^0(\mathcal{O}(D)) - \dim H^0(\Omega(-D)) = \deg D + 1 - g$.

Let's take a look at that quantity g. Take $K = \mathrm{div}(\omega)$, where $\omega \in \mathrm{Frac}(\Omega(X)) \setminus \{0\}$. The map $H^0(\mathcal{O}(K - D)) \to H^0(\Omega(-D))$ given by $f \mapsto f\omega$ is an isomorphism. If Riemann-Roch holds, then $\dim H^0(\mathcal{O}(D)) - \dim H^0(\mathcal{O}(K - D)) = \deg D - g + 1$ and $\dim H^0(\mathcal{O}(K - D)) - \dim H^0(\mathcal{O}(D)) = \deg(K - D) - g + 1$. Adding the two equations gives $0 = \deg D + \deg(K - D) - 2g + 2$, so $\deg K = 2g - 2$. Therefore if Riemann-Roch holds for some g, then g is the familiar genus.

Put $D = 0$, so $1 - \dim H^0(\Omega(D)) = 1 - g$, giving $\dim H^0(\Omega(D)) = g$. We have finally proved that g is a nonnegative integer.

As an example, we asked, what is a genus 0 curve? Let X be a connected smooth projective curve of genus 0. Let $p \in X$. Then $\dim H^0(\mathcal{O}(p)) \geq \deg(p) + 1 - g = 2$, so there is a nonconstant $f \in H^0(\mathcal{O}(p))$. Now $f : X \to \mathbb{P}^1$ has $f^{-1}(\infty) = \{p\}$ with multiplicity 1, so $X \cong \mathbb{P}^1$.

What about $g = 1$? Let X be genus 1 and $p \in X$. We have $\dim H^0(\mathcal{O}(2p)) \geq 2$, so there is a degree 2 map $f : X \to \mathbb{P}^1$ satisfying $f^{-1}(\infty) = \{p\}$ with multiplicity 2. We see that f is ramified at 4 points, one of which is p, and that $X \setminus \{p\}$ looks like the elliptic curve $y^2 = (x - a)(x - b)(x - c)$, where a, b, c are the other points where f is ramified.

What about $g = 2$? Let K be a canonical divisor with $K = \mathrm{div}(\omega)$, so $\dim K = 2$. We've shown that $\dim H^0(\Omega(K)) = 2$ (i.e., $\dim H^0(\Omega(K)) = \deg K + 1 - g + \dim H^0(\Omega(-K))$). Let ω_1, ω_2 be a basis for $H^0(\Omega(K))$. Take $f = \frac{\omega_1}{\omega_2}$, so that $f : X \to \mathbb{P}^1$ is degree 2 with $\#f^{-1}(\infty) = \#\{\omega_2 = 0\} = 2$. By similar reasoning as above, the curve looks like

$$y^2 = (x - a)(x - b)(x - c)(x - d)(x - e),$$

with branch points a, b, c, d, e, ∞.

What about $g = 3$? The same argument as before gives $\dim H^0(\mathcal{O}(K)) = 3$. Fixing a basis $\omega_1, \omega_2, \omega_3$, we look at the map $X \to \mathbb{P}^2$ given by $x \mapsto (\omega_1(x) : \omega_2(x) : \omega_3(x))$. The image of this map is either smooth of degree 4, or is a two-fold cover of a conic.

December 1

Today we're taking the first step towards the proof of the Riemann-Roch Theorem:

Theorem (Riemann-Roch). *Suppose X is an irreducible smooth projective curve of genus g, and D is a divisor on it, then*

$$\dim H^0(\Omega(D)) - \dim H^0(\Omega(-D)) = \deg D - g + 1$$

Let's consider a simple case: $D = z_1 + \cdots + z_d$, where d_i are distinct points on X. We've given an inductively proof of the following fact:

$$\dim H^0(\Omega(D)) \leqslant \deg D + 1 = d + 1$$

If we unpack the inductive proof, we'll see what is going on: if two functions f and g in $H^0(\Omega(D))$ have the same coefficients on their poles at $z_1, \cdots, z_d$, then $f - g$ is a constant, so that we can get the following exact sequence:

$$0 \longrightarrow k \longrightarrow H^0(\Omega(D)) \longrightarrow k^d,$$

in which the last map is obtained by taking the coefficient at each pole. Notice that a function on X has poles nowhere is a constant function, so the kernel is k. From the exact sequence we see that $\dim H^0(\Omega(D))$ is at most $d + 1$.

For any rational 1-form η on X, we have

$$\sum_{x \in X} \mathrm{res}_x \omega = 0.$$

So if $f \in H^0(\Omega(D))$ and $\omega \in H^0(\Omega)$, we'll get:

$$0 = \sum_{x \in X} \mathrm{res}_x(f\omega) = \sum_{i=1}^{d} \mathrm{res}_{z_i}(f\omega)_{z_i},$$

which provides a linear constraint on the image of $H^0(\Omega(D))$ in k^d. If we take $g = \dim H^0(\Omega)$, then this suggests that $\dim H^0(\Omega(D))$ might be $d + 1 - g$, as we impose g linear constraints. More carefully, we actually get constraints from $H^0(\Omega)/H^0(\Omega(-D))$, whose dimension is equal to $g - \dim H^0(\Omega(-D))$. Suppose $D \geqslant 0$, then we'll get:

$$\dim H^0(\Omega(D)) \leqslant 1 + (d - (g - \dim H^0(\Omega(-D)))).$$

Thus the Riemann-Roch Theorem basically says that the only obstacle to a rational function having specified pole expansion is that $\sum_{x \in X} \mathrm{res}_x(f\omega) = 0$ for any rational 1-form w on X.

We're going to prove the following theorem:

Theorem (Approximate Riemann-Roch Theorem).

$$\dim H^0(\Omega(D)) = \max(\deg D, 0) + O(1).$$

We've already proved the upper bound $\dim H^0\mathcal{O}(D)) \leq \deg D + 1$, and it is obvious that $H^0(\mathcal{O}(D)) = 0$ if $\deg D \leq 0$, so it remains to show that there exist some constant C, depending only on X, such that $\dim H^0(\Omega(D)) \geqslant \deg D - C$. Take an embedding $X \hookrightarrow \mathbb{P}^N$, and let D_∞ be the divisor supported on the points of $X \cap \{z_0 = 0\}$, where the multiplicity for each point equals its ramification. Take $D = tD_\infty$, we have the following fact:

$$\dim H^0(\Omega(tD_\infty)) \geqslant t \deg D_\infty - C,$$

for some constant C independent of t. This is because any inhomogeneous polynomial of degree t in $\frac{x_1}{x_0}, \cdots, \frac{x_N}{x_0}$ lies in $H^0(\Omega(tD_\infty))$, and

$$\begin{aligned}\dim H^0(\Omega(tD_\infty)) &\geqslant h^{\text{func}}(t)\\ &= h^{\text{poly}}(t) \qquad (\text{for } t \gg 0)\\ &= t \deg X - C\\ &= t \deg D_\infty - C.\end{aligned}$$

Remark. Another way to think about Riemann-Roch is that it provides the constant term for the Hilbert polynomial, i.e. $h^{\text{poly}}(t) = (\deg X)t - g + 1$.

We have the following corollary:

Corollary. *There exist a constant C_3 such that any divisor D is rationally equivalent to a divisor of the form $tD_\infty + E$, where*

$$\sum_{x \in X} |E(x)| \leq C_3.$$

Now we can prove the Approximate Riemann-Roch Theorem as follows:

Proof. Let D be rationally equivalent to $tD_\infty + E$ as above, we have:

$$H^0(\Omega(D)) \cong H^0(\Omega(tD_\infty + E).$$

Then it gives:

$$\begin{aligned}\dim H^0(\Omega(tD_\infty + E) &\geqslant \dim H^0(\Omega(tD_\infty)) - \sum_{x \in X} \max(-E(x), 0)\\ &\geqslant t \deg D_\infty - C_2 - C_3\\ &\geqslant \deg D - C_4.\end{aligned}$$

□

December 3

Announcements: The Baby Student Algebraic Geometry Seminar will be 5-6pm on Mondays next semester, organized by Francesca, in room 4096 of East Hall.

Definition. *For $W \subset X$, then $\mathcal{O}(D)_W = \{f \in \mathrm{Frac}(X) : v_x(f) + D(X) \geq 0,\ x \in W\}$.*

For example, if $W = X$, $\mathcal{O}(D)_X = H^0(\mathcal{O}(D))$. Notice that $\mathcal{O}_W = \mathcal{O}(0)_W$. We define $\Omega(D)_W$ similarly.

Suppose $X = U \cup V$ with U, V nonempty open sets neither of which is X. Then $H^0(\mathcal{O}(D))$ is the kernel of the map $\mathcal{O}(D)_U \oplus \mathcal{O}(D)_V \to \mathcal{O}(D)_{U\cap V}$ where $(f, g) \mapsto f - g$.

Then we have the exact sequence

$$0 \to H^0(\mathcal{O}(D)) \to \mathcal{O}(D)_U \oplus \mathcal{O}(D)_V \to \mathcal{O}(D)_{U \cap V}$$

which we can extend by defining $H^1(\mathcal{O}(D); U, V)$ to be the cokernel of the map $\mathcal{O}(D)_U \oplus \mathcal{O}(D)_V \to \mathcal{O}(D)_{U \cap V}$ so that

$$0 \to H^0(\mathcal{O}(D)) \to \mathcal{O}(D)_U \oplus \mathcal{O}(D)_V \to \mathcal{O}(D)_{U \cap V} \to H^1(\mathcal{O}(D); U, V) \to 0$$

is exact.

We want to show that this does not depend on the choice of open sets U, V, so we introduce the following claim:

Proposition. *Let $q \in U \cap V$. Then $H^1(\mathcal{O}(D); U, V) \cong H^1(\mathcal{O}(D); U, V \setminus \{q\})$.*

Proof Sketch. We consider the following commutative diagram, with exact rows, and apply the Snake lemma:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(D)_U \oplus \mathcal{O}(D)_V \arrow[r] \arrow[d] & \mathcal{O}(D)_U \oplus \mathcal{O}(D)_{V \setminus \{q\}} \arrow[r] \arrow[d] & \mathcal{O}(D)_{V \setminus \{q\}}/\mathcal{O}(D)_V \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathcal{O}(D)_{U \cap V} \arrow[r] & \mathcal{O}(D)_{U' \cap V} \arrow[r] & \mathcal{O}(D)_{(U \cap V) \setminus \{q\}}/\mathcal{O}(D)_{U \cap V} \arrow[r] & 0
\end{tikzcd}$$

Then by the snake lemma, we have that

$$0 \to H^0(\mathcal{O}(D)) \to H^0(\mathcal{O}(D)) \to 0 \to H^1(\mathcal{O}(D); U, V) \to H^1(\mathcal{O}(D); U', V) \to 0 \to 0.$$

is exact and thus these H^1 are isomorphic.

□

The point being omitted from the sketch here is why the right vertical map is an isomorphism. In the commutative diagram above, if $V \subset X$, which is not empty or X, then the map $\mathcal{O}(D)_V \to \mathcal{O}(D)_{V \setminus q}$ has cokernel isomorphic to $\mathrm{Frac}(X)/\mathcal{O}(D)_{\{q\}}$ which is the set of all rational functions on X quotiented by the set of functions vanishing to order greater than or equal to $-D(q)$ at q. In particular, $\mathrm{Frac}(X)/\mathcal{O}(D)_{\{q\}}$ is depends only on q, not on the open set V around it. Identifying the vector spaces in the right hand column with $\mathrm{Frac}(X)/\mathcal{O}(D)_{\{q\}}$ in this way, the vertical map is the identity. See the notes for more detail.

We also talked about an alternative definition of cohomology. If we pick finitely many points $z_1, \ldots, z_r \in X$ and let $V = X \setminus \{z_1, \ldots, z_r\}$ then we have an exact sequence

$$0 \to H^0(\mathcal{O}(D)) \to \mathcal{O}(D)_V \oplus \left(\oplus_i \mathcal{O}(D)_{\{z_i\}}\right) \to \mathrm{Frac}(X)^{\oplus r} \to H^1(\mathcal{O}(D)) \to 0$$

where we define $H^1(\mathcal{O}(D))$ to be the cokernel of the map $\mathcal{O}(D)_V \oplus \left(\oplus_i \mathcal{O}(D)_{\{z_i\}}\right) \to \mathrm{Frac}(X)^{\oplus r}$ to make the sequence exact. We would need to show that it does not depend on how you pick the z_i. Next term, we will talk about homological algebra, which explain why so many different maps whose kernel is H^0 all have cokerel H^1

Back to our previous proposition, we have proved that, using this proposition repeatedly, we get natural maps $H^1(\mathcal{O}(D); U_1, V_1) \to H^1(\mathcal{O}(D); U_2, V_2)$ which are isomorphisms for any two covers, U_1, V_1 and U_2, V_2.

December 5

Recall our setting: X is a smooth connected projective curve, and we have an open cover $X = U \cup V$ (where $U, V \neq \varnothing, X$). Then we have an exact sequence:

$$0 \to H^0(\Omega(D)) \to \Omega(D)_U \oplus \Omega(D)_V \to \Omega(D)_{U\cap V} \to H^1(\Omega(D); U, V) \to 0$$

and we define $H^1(\Omega(D))$ similarly.

An example: let $X = \mathbb{P}^1$. We can write $\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^1$, where the first factor has coordinate t and the second has coordinate $u = t^{-1}$. We want to understand $\Omega_U \oplus \Omega_V \to \Omega_{U\cap V}$. This is like $k[t]dt \oplus k[t^{-1}]dt^{-1} \to k[t, t^{-1}]dt$. So the exact sequence looks like:

$$0 \to k[t]dt \oplus t^{-2}k[t^{-1}]dt \to k[t, t^{-1}]dt \to k \cdot \frac{dt}{t} \to 0$$

Today: let's start with a Noether normalization $\pi : X \to \mathbb{P}^1$, where X is as before (a smooth connected projective curve). Let D_∞ be the divisor $\pi^{-1}(\infty)$ with multiplicity. If $t \gg 0$, then $H_1(\Omega(tD_\infty)) = 0$. To see this, cover $\mathbb{P}^1$ by $\mathbb{P}^1 \setminus \infty$ and $\mathbb{P}^1 \setminus 0$. Write z for the variable on $\mathbb{P}^1$, and call these open sets U, V. We want to understand: $\Omega(tD_\infty)_U \oplus \Omega(tD_\infty)_V \to \Omega(tD_\infty)_{U\cap V}$. This is like $\Omega_U \oplus z^t\Omega_V \to \Omega_{U\cap V}$.

Here, Ω_U is a free $k[z]$-module of rank $\deg \pi$, Ω_V is a free $k[z^{-1}]$-module of rank $\deg(\pi)$ and $\Omega_{U\cap V}$ is a free $k[z, z^{-1}]$-module of rank $\deg \pi$. This comes down to the fact that if $M, N \subset k[z, z^{-1}]^{\oplus r}$ are rank r $k[z]$- and $k[z^{-1}]$-submodules respctively, then $M + z^k N = k[z, z^{-1}]^{\oplus r}$ for $t \gg 0$. So we've shown that $H_1(\Omega(tD_\infty)) = 0$.

Now, the "incredibly useful exact sequence": let $D' = D + p$, where D is a divisor and p a point. We have a long exact sequence

$$0 \to H^0(\Omega(D)) \to H^0(\Omega(D)) \to k \to H^1(\Omega(D)) \to H^1(\Omega(D')) \to 0$$

The proof is by commutative diagram, and there are three cases (all equally as easy). For instance, we have the commutative diagram as below and if $p \in V \setminus U$ we get a snake:

$$\begin{tikzcd}
0 \arrow[r, dashed] & H^0(\Omega(D)) \arrow[r, dashed] \arrow[d] & H^0(\Omega(D')) \arrow[r, dashed] \arrow[d] & k \arrow[d] \arrow[dddll, dashed, rounded corners] & \\
0 \arrow[r] & \Omega(D)_U \oplus \Omega(D)_V \arrow[r] \arrow[d, "f"] & \Omega(D')_U \oplus \Omega(D')_V \arrow[r, "p"] \arrow[d, "g"] & k \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & \Omega(D)_{U\cap V} \arrow[r, "i"] \arrow[d] & \Omega(D')_{U\cap V} \arrow[r] \arrow[d] & 0 \arrow[d] & \\
 & H^1(\Omega(D)) \arrow[r, dashed] & H^1(\Omega(D')) \arrow[r, dashed] & 0 &
\end{tikzcd}$$

Corollary. $\dim H(D) < \infty$ *for any* D.

This follows from the above exact sequence by changing divisors from tD_∞ down to D by removing points. Another corollary is that:

Corollary. *If* $D \subset E$*, then we have a surjection* $H^1(\Omega(D)) \to H^1(\Omega(E))$*.*

Notice that this last statement is dual to what we have for H^0, suggesting that we should be on the lookout for some sort of duality. We are in a position to prove:

Theorem ("Homological Riemann-Roch")**.** There is an integer h so that $\dim H^0(\Omega(D)) - \dim H^1(\Omega(D)) = \deg D - h + 1$.

This follows immediately from the long exact sequence, since we have $\dim H^0(\Omega(D)) - \dim H^1(\Omega(D')) = \dim H^0(\Omega(D)) - \dim H^1(\Omega(D)) + 1$. We'd be done with Riemann-Roch if we knew $\dim H^0(\Omega(D)) = \dim H^1(\Omega(-D))$ (or $\dim H^0(\Omega(D)) = \dim H^1(\Omega(-D))$. For this we need:

Theorem (Serre duality)**.** There is a perfect pairing $H^0(\Omega(D)) \times H^1(\Omega(-D)) \to k$ (or $H^0(\Omega(D)) \times H^1(\Omega(-D)) \to k$.

December 8

We recalled the end of the previous class: Let X be a smooth projective connected and hence irreducible curve. $X = U \bigcup V$ is an open cover with $U, V \neq \varnothing, X$.

$$H^1(\Omega(D)) := \mathrm{Coker}(\Omega(D)_U \oplus \Omega(D)_V \to \Omega(D)_{U\cap V})$$

If $D' = D + p$, then we have an exact sequence

$$0 \longrightarrow H^0(\Omega(D)) \longrightarrow H^0(\Omega(D')) \longrightarrow k \longrightarrow H^1(\Omega(D)) \longrightarrow H^1(\Omega(D')) \longrightarrow 0\ .$$

From this we deduce:

$$\mathrm{dim} H^0(\Omega(D)) - \mathrm{dim} H^1(\Omega(D)) = \mathrm{deg} D - h + 1$$

where $h = \mathrm{dim} H^1(\Omega)$. In order to deduce Riemann-Roch from this, we need to know that $\dim H^0(\mathcal{O}(D)) = \dim H^1(\Omega(-D))$.

More precisely, we will prove:

Serre duality There is a perfect pairing $H^1(\Omega(D)) \times H^0(\Omega(-D)) \to k$.

Equivalently, there is a pairing $H^0(\Omega(D)) \times H^1(\Omega(-D)) \to k$.

Today, we describe this pairing.

Proposition If ω is a nonzero rational $1-$form on X, then $\sum_{x\in X} \mathrm{res}_x \omega = 0$ where X is a smooth projective curve.

Proof: Hint: char $k = 0$, it suffices to prove on $\mathbb{P}^1$.Then we build a map $\int : H^1(\Omega) \to k$, where $H^1(\Omega) := \mathrm{Coker}(\Omega_U \oplus \Omega_V \to \Omega_{U\cap V})$. Let $\omega \in \Omega_{U\cap V}$, then $\int \omega = \sum_{x\in X-U} \mathrm{res}_x \omega$.

In general, let $f \in H^0(\mathrm{O}(D))$, $\omega \in \Omega(-D)_{U\cap V}$, representing a class $[\omega]$ in $H^1(\Omega(-D))$. The Serre duality pairing is $< f, [\omega] >= \int f\omega$. This pairing depends only on the class $< f, [\omega] >$.

December 10

We are ready to finish the proof of Serre duality, and thus of Riemann-Roch. We will need the following facts from our earlier work on Asymptotic Riemann-Roch:

- If $\deg D$ is negative, then $H^0(\mathcal{O}(D)) = 0$.
- If $\deg D$ is sufficiently large, then $H^1(\mathcal{O}(D)) = 0$.

Our proof is based on chasing commutative diagrams. If $D' = D + p$, then we have an exact sequence

$$0 \to H^0(\mathcal{O}(D)) \to H^0(\mathcal{O}(D')) \to k \to H^1(\mathcal{O}(D)) \to H^1(\mathcal{O}(D')) \to 0 .$$

Now, $-D = -D' + p$, so we have a second long exact sequence

$$0 \leftarrow H^0(\Omega(-D)) \leftarrow H^0(\Omega(-D')) \leftarrow k \leftarrow H^1(\Omega(-D)) \leftarrow H^1(\Omega(-D')) \leftarrow 0$$

which we can dualize to

$$0 \to H^0(\Omega(-D))^\vee \to H^0(\Omega(-D'))^\vee \to k \to H^1(\Omega(-D))^\vee \to H^1(\Omega(-D'))^\vee \to 0 .$$

The Serre pairing gives us vertical maps

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{O}(D)) \arrow[r] \arrow[d, "\sigma"] & H^0(\mathcal{O}(D')) \arrow[r, "\alpha"] \arrow[d, "\sigma'"] & k \arrow[r] \arrow[d, equal] & H^1(\mathcal{O}(D)) \arrow[r] \arrow[d] & H^1(\mathcal{O}(D')) \arrow[r] \arrow[d] & 0 . \\
0 \arrow[r] & H^0(\Omega(-D))^\vee \arrow[r] & H^0(\Omega(-D'))^\vee \arrow[r, "\beta"] & k \arrow[r] & H^1(\Omega(-D))^\vee \arrow[r] & H^1(\Omega(-D'))^\vee \arrow[r] & 0
\end{tikzcd}$$

These form a commutative diagram. We want to prove the vertical maps are isomorphisms.

If $\deg D \ll 0$, then columns 1 and 2 are zero, so σ and σ' are trivially isomorphisms.

If $\deg D \gg 0$, then columns 4 and 5 are zero, and the third column is an isomorphism, so $\mathrm{Ker}(\sigma) \cong \mathrm{Ker}(\sigma')$ and $\mathrm{CoKer}(\sigma) \cong \mathrm{CoKer}(\sigma')$.

So, as D increases from very negative to very positive, the kernel and cokernel of σ start at 0 and eventually stabilize at some finite dimensional vector space. Of course, we actually want to show they are 0 for all D.

We split off the left three columns to make a 2×3 diagram: The Serre pairing gives us vertical maps

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{O}(D)) \arrow[r] \arrow[d, "\sigma"] & H^0(\mathcal{O}(D')) \arrow[r, "\alpha"] \arrow[d, "\sigma'"] & \mathrm{Im}(\alpha) \arrow[r] \arrow[d, hook] & 0 . \\
0 \arrow[r] & H^0(\Omega(-D))^\vee \arrow[r] & H^0(\Omega(-D'))^\vee \arrow[r, "\beta"] & \mathrm{Im}(\beta) \arrow[r] & 0
\end{tikzcd}$$

Note that $\mathrm{Im}(\alpha)$ is literally a subset of $\mathrm{Im}(\beta)$. From the snake lemma, we get

$$\mathrm{Ker}(\sigma) \cong \mathrm{Ker}(\sigma') \text{ and } \mathrm{CoKer}(\sigma) \subseteq \mathrm{CoKer}(\sigma').$$

Using the result on kernels over and over, we deduce that $\mathrm{Ker}(H^0(\mathcal{O}(D)) \longrightarrow H^1(\Omega(-D))^\vee)$ is independent of D. It is trivially 0 when $\deg D \ll 0$ (since the two spaces are 0). So the kernel vanishes for all D.

We now turn to the cokernels. Set

$$q(D) := \mathrm{CoKer}(H^0(\mathcal{O}(D)) \longrightarrow H^1(\Omega(-D))^\vee).$$

What we know so far is that

(1) $q(D) = 0$ for $\deg D \ll 0$.
(2) If $D \leq E$ then $q(D) \subseteq q(E)$.
(3) As $\deg D$ gets large, $q(D)$ stays bounded.

From point (2), it is enough to show that $q(tD_\infty)$ vanishes for t large, since every D is $< tD_\infty$ for t large enough. Let $M = \bigcup_{t=0}^\infty q(tD_\infty)$. From point (3), M is a finite dimensional k-vector space.

We have a collection of injections:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] \\
0 \arrow[r] & H^0(\mathcal{O}(D_\infty)) \arrow[r, hook] \arrow[d] & H^0(\mathcal{O}(2D_\infty)) \arrow[r, hook] \arrow[d] & H^0(\mathcal{O}(3D_\infty)) \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & \bigcup_{t=0}^{\infty} H^0(\mathcal{O}(tD_\infty)) = \mathcal{O}_U \arrow[d] \\
0 \arrow[r] & H^0(\mathcal{O}(D_\infty)) \arrow[r, hook] \arrow[d] & H^0(\mathcal{O}(2D_\infty)) \arrow[r, hook] \arrow[d] & H^0(\mathcal{O}(3D_\infty)) \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & \bigcup_{t=0}^{\infty} H^1(\Omega(-tD_\infty))^\vee \arrow[d] \\
0 \arrow[r] & q(D_\infty) \arrow[r, hook] \arrow[d] & q(2D_\infty) \arrow[r, hook] \arrow[d] & q(3D_\infty) \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & \bigcup_{t=0}^{\infty} q(tD_\infty) = M \arrow[d] \\
 & 0 & 0 & 0 & & 0
\end{tikzcd}$$

Let U be the affine open $X \setminus \pi^{-1}(\infty)$. In the rightmost column, $\bigcup_{t=0}^{\infty} H^0(\mathcal{O}(tD_\infty)) = \mathcal{O}_U$. When we have finished the proof, we will know that $\bigcup_{t=0}^{\infty} H^1(\Omega(-tD_\infty))^\vee$ is isomorphic to $\mathcal{O}_U$. We can see directly that $\bigcup_{t=0}^{\infty} H^1(\Omega(-tD_\infty))^\vee$ is a $\mathcal{O}_U$ module and that the vertical map is a map of $\mathcal{O}_U$ modules.

So M is a $\mathcal{O}_U$ module which is finite dimensional as a k-vector space. In other words, M is a torsion module. Let $p \in \mathcal{O}_U$ annihilate M.

Now, run the whole argument again with D_∞ replaced by $D := D_\infty \cup Z(p)$. Let $N = \bigcup_{t=0}^{\infty} q(tD)$. Since $tD_\infty \leq tD$, we see that M embeds into N and one can check that this is a map of $\mathcal{O}_U$ modules. But N is a $\mathcal{O}_{U \setminus Z(p)}$ module and $p^{-1} \in \mathcal{O}_{U \setminus Z(p)}$. So p acts by 0 on M, and acts invertibly on N. This shows M is zero, concluding the proof.