Problem Set One: Due Thursday, January 25

See the course website for policy on collaboration. All references to FOAG refer to the July 31, 2023 edition of *Foundations of Algebraic Geometry*, by Ravi Vakil.

Problems from the textbook:

Exercise 4.1.D: The construction of a sheaf $\widetilde{M}$ from a module M. Feel free to just verify that $\widetilde{M}$ is a sheaf on the affine base, without spelling out why that implies that it gives a sheaf on $\operatorname{Spec} A$ and without checking that $\widetilde{M}$ is an $\mathcal{O}_{\operatorname{Spec} A}$-module.
Exercise 4.1.G: The full subcategory of quasi-coherent sheaves is equivalent to the category of modules on an affine scheme.
Exercise 6.3.A: Exact sequences of quasi-coherent sheaves are exact on every affine open. Contrast this result with Problem 2.

Problem 1. Let k be a field and let $\mathbb{A}^n = \operatorname{Spec} k[x_1, x_2, \ldots, x_n]$. Conventionally, we think of $\mathbb{A}^n$ as "n-dimensional space". In each part of this problem, I'll describe a geometric map involving $\mathbb{A}^n$ in informal terms and ask you to give the corresponding map of rings.

(1) "Vector addition" is a map $\mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n$. What is the corresponding map $k[x_1, \ldots, x_n] \to k[y_1, \ldots, y_n] \otimes_k k[z_1, \ldots, z_n]$?
(2) "Scalar multiplication" is a map $\mathbb{A}^1 \times \mathbb{A}^n \to \mathbb{A}^n$. What is the corresponding map $k[x_1, \ldots, x_n] \to k[t] \otimes_k k[y_1, \ldots, y_n]$?

Problem 2. Let k be a field and let $\mathbb{A}^2$ be $\operatorname{Spec} k[x, y]$. Let $(0, 0)$ be the point of $\mathbb{A}^2$ corresponding to the maximal ideal $\langle x, y\rangle$ of $k[x, y]$ and let U be the open set $\mathbb{A}^2 \setminus \{(0, 0)\}$.

Let ϕ be the map $k[x, y]^{\oplus 2} \to k[x, y]$ given by $\phi\left(\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]\right) = xf + yg$; let $\widetilde{\phi}$ be the induced map of sheaves $\mathcal{O}_{\mathbb{A}^2}^{\oplus 2} \to \mathcal{O}_{\mathbb{A}^2}$ on $\mathbb{A}^2$.

(1) Is the map of modules $\phi : k[x, y]^{\oplus 2} \to k[x, y]$ surjective?
(2) Is the map of sheaves $\widetilde{\phi} : \mathcal{O}_{\mathbb{A}^2}^{\oplus 2} \to \mathcal{O}_{\mathbb{A}^2}$ surjective on $\mathbb{A}^2$?
(3) Is the map of sheaves $\widetilde{\phi} : \mathcal{O}_U^{\oplus 2} \to \mathcal{O}_U$ surjective on U? This is the restriction of $\widetilde{\phi}$ to the open set U.
(4) Is the map of global sections, $\widetilde{\phi}(U) : \mathcal{O}(U)^{\oplus 2} \to \mathcal{O}(U)$, surjective?

Problem 3. Let R be an integral domain and let J be an ideal of R. The ideal J is called ***principal*** if $J = fR$ for some nonzero $f \in R$. The ideal J is called locally principal if there are localizations R_{f_1}, R_{f_2}, $\ldots$, R_{f_k} with $\operatorname{Spec} R = \bigcup_j \operatorname{Spec} R_{f_j}$ such that J_{f_i} is principal in R_{f_i} for each i. In this exercise, we'll see an example of a locally principal, but non-principal, ideal, which is surprisingly geometric.

Let $R = \mathbb{R}[x, y]/\langle x^2 + y^2 - 1\rangle$. Let J be the ideal $\langle x - 1, y\rangle$.

(1) Verify that J is locally principal.
(2) Verify that J is not principal. There are several ways to do this, but one way is to think about how the function $f(x, y)$ could behave on the circle $x^2 + y^2 = 1$, if we had $J = f(x, y)R$.