PROBLEM SET TWO: DUE THURSDAY, FEBRUARY 1

See the course website for policy on collaboration. All references to FOAG refer to the July 31, 2023 edition of *Foundations of Algebraic Geometry*, by Ravi Vakil.

Problems from the textbook:

14.2.B This exercise is "essential", but it is also straightforward. It tells you that pullbacks of vector bundles are computed in a completely straightforward way.
14.2.D This exercise shows you how to represent dual vector bundles in coordinates. It is a good example to make you do a concrete computation.
14.2.F Tensoring with a locally free sheaf preserves exactness. This comes up all the time.
14.2.H The definition of the Picard group, an important object in algebraic geometry.
14.3.G This exercise shows how to write down a short exact sequence of vector bundles in coordinates. You'll get more practice with it in Problem 2.

Problem 1. As described in Section 14.2 of *FOAG*, to give a vector bundle of rank r on a scheme X is to give an open cover U_i of X and transition matrices $T_{ij} \in \mathrm{GL}_r(\mathcal{O}(U_i \cap U_j))$ obeying[1] $T_{ik}|_{U_i \cap U_j \cap U_k} = T_{jk}|_{U_i \cap U_j \cap U_k} \circ T_{ij}|_{U_i \cap U_j \cap U_k}$. Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles of ranks p and q on X. Let U_i be an open cover of X on which both $\mathcal{E}$ and $\mathcal{F}$ are trivial, with transition matrices P_{ij} and Q_{ij}.

(1) Show that $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$ is isomorphic to the set of collections of matrices $H_i \in \mathrm{Mat}_{q \times p}(\mathcal{O}(U_i))$ such that $H_j|_{U_i \cap U_j} \circ P_{ij} = Q_{ij} \circ H_i|_{U_i \cap U_j}$.
(2) Show that $\mathcal{E} \cong \mathcal{F}$ if and only if $p = q$ and there are invertible matrices $G_i \in \mathrm{GL}_p(\mathcal{O}(U_i))$ such that $Q_{ij} = G_j|_{U_i \cap U_j} \circ P_{ij} \circ G_i^{-1}|_{U_i \cap U_j}$.

Problem 2. In this problem, we work with some basic vector bundles on $\mathbb{P}^1$. Let k be a field. Cover $\mathbb{P}^1$ with two open charts: $U = \operatorname{Spec} k[t]$ and $V = \operatorname{Spec} k[t^{-1}]$, so $U \cap V = \operatorname{Spec} k[t, t^{-1}]$. The line bundle $\mathcal{O}(n)$ is formed by taking a section σ of $\mathcal{O}_U$ and gluing it $t^{-n}\sigma$ in $\mathcal{O}_V$. So the global sections of $\mathcal{O}(n)$ are ordered pairs (f, g) with $f \in k[t]$ and $g \in k[t^{-1}]$ such that $g = t^{-n} f$. Let x_1 and x_2 be the sections $(1, t^{-1})$ and $(t, 1)$ of $\mathcal{O}(1)$.

(1) Define $\beta : \mathcal{O}(0)^{\oplus 2}$ to $\mathcal{O}(1)$ by $(f_1, f_2) \mapsto f_1 x_1 + f_2 x_2$. Show that β is a surjective map of sheaves.
(2) Show that the kernel of β is isomorphic to $\mathcal{O}(-1)$. So we have a short exact sequence $0 \to \mathcal{O}(-1) \xrightarrow{\alpha} \mathcal{O}^{\oplus 2} \xrightarrow{\beta} \mathcal{O}(1) \to 0$.
(3) Encode the vector bundles $\mathcal{O}(-1)$, $\mathcal{O}^{\oplus 2}$ and $\mathcal{O}(1)$ with the matrices $\begin{bmatrix} t \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} t^{-1} \end{bmatrix}$ respectively. Give the matrices in $\mathrm{Mat}_{1 \times 2}(k[t])$ and $\mathrm{Mat}_{1 \times 2}(k[t^{-1}])$ representing β and the matrices in $\mathrm{Mat}_{2 \times 1}(k[t])$ and $\mathrm{Mat}_{2 \times 1}(k[t^{-1}])$ representing α.
(4) From **14.3.G**, there must be a matrix of the form $\begin{bmatrix} t^{-1} & * \\ 0 & t \end{bmatrix}$ representing $\mathcal{O}^{\oplus 2}$. From Problem 1, we should be able to factor

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = G_1 \begin{bmatrix} t^{-1} & * \\ 0 & t \end{bmatrix} G_2^{-1}$$

for $G_1 \in \mathrm{GL}_2(k[t])$ and $G_2 \in \mathrm{GL}_2(k[t^{-1}])$. Give the matrices $\begin{bmatrix} t^{-1} & * \\ 0 & t \end{bmatrix}$, G_1 and G_2.

[1] If I were writing the book, I would switch the order of the subscripts in T_{ij}, but I will follow the book.