Problem Set Three: Due Thursday, February 8

See the course website for policy on collaboration. All references to FOAG refer to the July 31, 2023 edition of *Foundations of Algebraic Geometry*, by Ravi Vakil.

Problems from the textbook: For all of these problems, feel free to assume that your scheme is noetherian, so that you don't have to think about the distinction between "finitely generated", "finitely presented" and "coherent".

Exercise 14.4.F: This shows that "locally free" is equivalent to "free stalks".
Exercise 14.4.K This shows that dimension of fibers only jumps up, not down, when we approach special points.
Exercise 14.4.L This justifies an assertion in class; that vector bundles are, in a certain sense, characterized by having fibers of constant dimension.

Exercise 14.7.C This is an excellent exercise to understand the sheaf corresponding to a graded module.
Exercise 14.7.D Compare the result of this exercise to problem 1.

Exercise 15.1.C This computes that the sections of $\mathcal{O}(d)$ on $\mathbb{P}^{n-1}$ are precisely the degree d polynomials in n variables; in class we saw why such polynomials give sections but didn't see why all sections were of this form.
Exercise 15.1.E This exercise classifies line bundles on $\mathbb{P}^1$. If you want to do $\mathbb{P}^{n-1}$ by brute force, this is also possible, but you'll have better tools for this problem after you've read Section 15.4.

Problem 1. This Problem gives you practice in computing the global sections of a sheaf on a projective scheme. Throughout, let k be a field. We identify $\mathbb{P}^{n-1}$ with $\operatorname{Proj} k[x_1, x_2, \ldots, x_n]$.

(1) Let X be the subscheme of $\mathbb{P}^1$ given by the homogenous ideal $\langle x_1 x_2 \rangle$. What are the global sections of $\mathcal{O}$ on X?
(2) Let Y be the subscheme of $\mathbb{P}^3$ given by the homogenous ideal $\langle x_1x_3,\ x_1x_4,\ x_2x_3,\ x_2x_4 \rangle$. What are the global sections of $\mathcal{O}$ on X?
(3) Let Z be the subscheme of $\mathbb{P}^3$ given by the homogenous ideal $\langle x_1^2x_3 - x_2^3,\ x_2x_4^2 - x_3^3,\ x_1x_4 - x_2x_3 \rangle$. What are the global sections of $\mathcal{O}(1)$ on X? Hint: First that Z is covered by the affine opens $Z \cap D_+(x_1)$ and $Z \cap D_+(x_4)$.