Problem Set Four: Due Thursday, February 15

See the course website for policy on collaboration. All references to FOAG refer to the February 6, 2024 edition of *Foundations of Algebraic Geometry*, by Ravi Vakil.

Problems from the textbook:
15.2.E and 15.2.G: I'm not sure why they are in this section of the book, but these are useful concrete facts to know about projective space, which are often helpful in computations.
15.4.A: This may be labeled "easy", but it is also important. I would do it by first thinking, for some divisor $D \subset \mathbb{P}^n$, about what the intersection $D \cap \mathbb{A}^n$ could look like.
15.4.H: ~~We finally learn that there are no line bundles on $\mathbb{P}^n$ other than $\mathcal{O}(d)$.~~ Removed because done in class.
15.5.C: This verifies that there is a section of $\mathcal{O}(D)$ which, indeed, has zero set D. It is "tricky" because there are a lot of parts to keep track of, not because the argument is long or surprising.

Problem 1. Let R be the ring $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$. In this problem, we'll compute the class group of R by hand. You may take for granted that R is a normal, one dimensional, Noetherian integral domain – also known as a Dedekind domain.

(1) Let $\mathfrak{m}$ be a maximal ideal of R. Show that $R/\mathfrak{m}$ is either isomorphic to $\mathbb{R}$ or to $\mathbb{C}$. We'll call $\mathfrak{m}$ "real" or "complex" accordingly.
(2) Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be two real maximal ideals of R. Show that $\mathfrak{m}_1\mathfrak{m}_2$ is principal. Remember to cover the case $\mathfrak{m}_1 = \mathfrak{m}_2$.
(3) Let $\mathfrak{m}$ be a complex maximal ideal of R. Show that there are real scalars a, b and c, not all 0, such that $ax + by + c \in \mathfrak{m}$. Show furthermore that $\mathfrak{m} = \langle ax + by + c \rangle$.
(4) Argue that Pic(Spec R) is cyclic of order 2. If the route sketched above doesn't work for you, you can also prove this by some other means; I tried to give the most direct attack I could.