Problem Set Five: Due Thursday, February 22

See the course website for policy on collaboration. All references to FOAG refer to the February 6, 2024 edition of *Foundations of Algebraic Geometry*, by Ravi Vakil.

Problems from the textbook:
15.4.J We carry out the verification, promised in class, of a divisor which is not locally principal.
15.4.M Here is a nice example of an explicit class group which is easy to compute by algebraic geometry, but which I think would be very hard by pure algebra.
15.4.Q The hints for this exercise aren't exactly the same as the way I'd do this. Feel free to prove that $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z}^2$ any way that you like.
16.1.C As the text says, this is "easy but important". It is a good example of the use of Nakayama's lemma.

Problem 1. Here are two problems on pulling back line bundles.

(1) Let $\alpha : \mathbb{P}^{n-1} \to \mathbb{P}^n$ be the inclusion of the hyperplane $x_{n+1} = 0$. Show that $\alpha^* \mathcal{O}_{\mathbb{P}^n}(d) \cong \mathcal{O}_{\mathbb{P}^{n-1}}(d)$.
(2) Let $\beta : \mathbb{P}^1 \to \mathbb{P}^r$ be the r-fold Veronese map; we can realize this in in homogeneous coordinates as $\beta(x : y) = (x^d : x^{d-1}y : \cdots : y^d)$. Show that $\beta^* \mathcal{O}_{\mathbb{P}^d}(1) \cong \mathcal{O}_{\mathbb{P}^1}(d)$. Hint: Writing $(z_0 : z_1 : \cdots : z_d)$ for the coordinates on $\mathbb{P}^d$, it may help to show that $\alpha(\mathbb{P}^1) \subset D_+(z_0) \cup D_+(z_d)$.

Problem 2. This problem gives an example where D is an effective locally principle divisor, but the restriction of $\mathcal{O}(D)$ to D is not effective.

Let k be a field and let $K = k(x, y)$. Let A and B be the subrings $k[x, x^{-1}y]$ and $k[xy^{-1}, y]$ of K.

(1) Verify that $A_{x^{-1}y} = B_{xy^{-1}}$ as subrings of K.

We define the scheme X to be the gluing of $\mathrm{Spec}(A)$ to $\mathrm{Spec}(B)$ along $\mathrm{Spec}(A_{x^{-1}y}) = \mathrm{Spec}(B_{xy^{-1}})$. We define the divisor D to be the closed subscheme where $D \cap \mathrm{Spec}(A) = V(x)$ and $D \cap \mathrm{Spec}(B) = V(y)$.

(2) Verify that $D \cong \mathbb{P}^1$.
(3) Let $\phi : D \to X$ be the inclusion of D into X. Verify that $\phi^* \mathcal{O}(D)$ is the line bundle $\mathcal{O}(-1)$ on D.

The variety X is called "the blowup of the affine plane at the origin." It is also (fun fact!) the total space of the line bundle $\mathcal{O}(-1)$.