See the course website for policy on collaboration. All references to FOAG refer to the February 21, 2024 edition of *Foundations of Algebraic Geometry*, by Ravi Vakil.

Problems from the textbook:

16.2.C, 16.2.E and 16.2.F We didn't get around to mentioning "very ample" in class; "very ample" means "pulled back from a projective embedding". But I think there are enough tools in the text for you to put it together.

Problem 1. This problem is rewritten because the old version was confusing to many. It is meant to be logically equivalent to the old one. Let X be a scheme and let D be an effective Cartier divisor, meaning a closed subscheme which is locally principal. Let $\mathcal{O}_D$ be the structure sheaf of D, considered as a sheaf on X. Show that there is a short exact sequence of sheaves

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

on X.

Problem 2. Here are some useful examples to illustrate subtleties of finiteness and degree:

(1) Let α be the obvious inclusion $\operatorname{Spec} k[x, x^{-1}] \to \operatorname{Spec} k[x]$. Show that all the fibers of α are finite, but that $k[x, x^{-1}]$ is not a finitely generated $k[x]$-module.
(2) Let β be the obvious morphism $\operatorname{Spec} k[t] \to \operatorname{Spec} k[t^2, t^3]$. Show that $k[t]$ is a finitely generated $k[t^2, t^3]$-module (so β is a finite map) but $k[t]$ is not a free $k[t^2, t^3]$-module. Hint: Compute the length of various fibers of β.

Problem 3. Here is an example of a curve which we will use over and over again – the hyperelliptic curve of genus g. Let k be a field, not of characteristic 2. Let $f(x) = f_{2g+1}x^{2g+1} + f_{2g}x^{2g} + \cdots + f_2x^2 + f_1x$ be a polynomial of degree $2g+1$ with $\operatorname{GCD}(f(x), f'(x)) = 1$.

We set

$$\begin{aligned} A_1 &= k[x_1, y_1]/\langle y_1^2 - (f_{2g+1}x_1^{2g+1} + f_{2g}x_1^{2g} + \cdots + f_2x_1^2 + f_1x_1) \\ A_2 &= k[x_2, y_2]/\langle y_2^2 - (f_1x_2^{2g+1} + f_2x_1^2 + \cdots + f_{2g}x_2^2 + f_{2g+1}x_2). \end{aligned}$$

(1) Check that $\operatorname{Spec} A_1$ and $\operatorname{Spec} A_2$ are smooth over k.
(2) Give an isomorphism $(A_1)_{x_1} \cong (A_2)_{x_2}$.

We define X to be the gluing of $\operatorname{Spec} A_1$ and $\operatorname{Spec} A_2$ along the isomorphism in (2).

(3) Check that the obvious maps $\operatorname{Spec}(A_1) \to \operatorname{Spec} k[x_1]$ and $\operatorname{Spec}(A_2) \to \operatorname{Spec} k[x_2]$ glue to give a finite map $\pi : X \to \mathbb{P}^1_k$.

We define p_i to be the point $\langle x_i, y_i \rangle$ in $\operatorname{Spec} A_i$.

(4) Check that $\pi^*(\mathcal{O}(1)) \cong \mathcal{O}(2p_1) \cong \mathcal{O}(2p_2)$.
(5) Compute the valuations $v_{p_2}(x_1)$, $v_{p_2}(y_1)$, $v_{p_2}(x_2)$, $v_{p_2}(y_2)$.
(6) For $d \leq 2g$, show that $\{1, x_1, x_1^2, \ldots, x_1^{\lfloor d/2 \rfloor}\}$ is a basis of $\Gamma(X, \mathcal{O}(dp_2))$. For $d \geq 2g+1$, show that $\{1, x_1, x_1^2, \ldots, x_1^{\lfloor d/2 \rfloor}, y_1, x_1y_1, \ldots, x_1^{\lfloor (d-2g-1)/2 \rfloor}\}$ is a basis for $\Gamma(X, \mathcal{O}(dp_2))$. Here $\lfloor z \rfloor$ means z rounded down to the nearest integer.