Problem Set Seven: Due Thursday, March 28

See the course website for policy on collaboration. All references to the textbook refer to the February 21, 2024 edition of *Foundations of Algebraic Geometry*, by Ravi Vakil.

Textbook problems:

18.2.A Check that the Čech complex is a complex.

18.2.D Check that sheaf cohomology commutes with localization. This is the key lemma which lets us define "relative sheaf cohomology".

Problem 1. This is a follow up to Problem 1 on the previous problem set. To summarize the results of that problem: Let X be a $\leq d$ dimensional subvariety of $\mathbb{P}^{n-1}$. Let λ_d be the rational map $\mathbb{P}^{n-1} \to \mathbb{P}^{d-1}$ given by $\lambda((x_1 : \cdots : x_n)) = (x_1 : \cdots : x_d)$, and let $Z_d = V_+(x_1, x_2, \ldots, x_d)$ be the locus where λ_d is not defined. You showed that, if $X \cap Z_d = \emptyset$, then $\pi : X \to \mathbb{P}^{d-1}$ is finite.

(1) Let k be an infinite field and let $X \subset \mathbb{P}^n_k$ be a closed subscheme of dimension $\leq d-1$. Show that we can choose coordinates in $\mathbb{P}^{n-1}_k$ such that $X \cap Z_d$ is empty.
(2) Let Y be a scheme and let $\mathfrak{p}$ be a point of Y with infinite residue field $\kappa(\mathfrak{p})$. Let X be a closed subscheme of $\mathbb{P}^{n-1}_Y$ such that the fiber $X \cap \mathbb{P}^{n-1}_{\mathfrak{p}}$ has dimension at most $d-1$. Show that we can find an open neighborhood $U \ni \mathfrak{p}$ and choose coordinates in $\mathbb{P}^{n-1}_U$ such that the map $\pi_d : X \cap \mathbb{P}^{n-1}_U \to \mathbb{P}^{d-1}_U$ is finite.
(3) In particular, let Y be a scheme such that $\kappa(\mathfrak{p})$ is infinite for every $\mathfrak{p} \in Y$ and let X be a closed subscheme of $\mathbb{P}^{n-1}_Y$ such that $X \cap \mathbb{P}^{n-1}_{\mathfrak{p}}$ is finite for every $\mathfrak{p} \in Y$. Show that the projection map $X \hookrightarrow \mathbb{P}^{n-1}_Y \to Y$ is finite.
(4) Let k be an infinite field, let X be a closed subscheme of $\mathbb{P}^{n-1}_k$, let Y be a separated k-scheme, and let $\pi : X \to Y$ be a morphism of k-schemes such that every fiber of π is finite. Show that π is a finite map. (Thanks to the several students who pointed out the requirement that Y be separated. Here is an example to demonstrate the issue. Let $X = \mathbb{P}^1$ and let Y be $\mathbb{P}^1$ with a point doubled; let $\pi : X \to Y$ be the open inclusion whose image misses one of the two doubled points. In a neighborhood of the missed point, this is not a finite map.)

Problem 2. Let $\mathcal{O}^*$ be the sheaf of abelian groups where $\mathcal{O}^*(U)$ is the unit group of the ring $\mathcal{O}(U)$. In this problem, we will relate the Čech cohomology of $\mathcal{O}^*$ to the Picard group. Note that $\mathcal{O}^*$ is a sheaf of abelian groups but **not** a quasi-coherent sheaf (or a sheaf of $\mathcal{O}_X$-modules at all!), so you'll have to work with definitions without much theory to aid you.

(1) Let U_i be an open cover of X, so we have a corresponding Čech complex
$$\prod_i \mathcal{O}^*(U_i) \xrightarrow{d^0} \prod_{i<j} \mathcal{O}^*(U_i \cap U_j) \xrightarrow{d^1} \prod_{i<j<k} \mathcal{O}^*(U_i \cap U_j \cap U_k).$$
Let $\phi_{ij} \in \operatorname{Ker}(d^1)$. Show that it makes sense to define a line bundle $\mathcal{L}(\phi)$ by gluing the trivial bundle over U_i to the trivial bundle over U_j by the gluing map ϕ_{ij}. Hint: Why does it matter that $\phi_{ij} \in \operatorname{Ker}(d^1)$?
(2) Suppose that ϕ and ψ are two classes in $\operatorname{Ker}(d^1)$. Show that the line bundles $\mathcal{L}(\phi)$ and $\mathcal{L}(\psi)$ are isomorphic if and only if $[\phi]$ and $[\psi]$ define the same class in the Čech cohomology group $H^1(X, \mathcal{O}^*, U_i)$. Conclude that we get an injection $H^1_{\mathcal{U}}(X, \mathcal{O}^*) \hookrightarrow \operatorname{Pic}(X)$.
(3) Show that $\operatorname{Pic}(X)$ is the union of the images of all the Čech groups $H^1_{\mathcal{U}}(X, \mathcal{O}^*)$, as $\mathcal{U}$ ranges over all open covers.
(4) In particular, suppose that U_i is an open cover where $U_i = \operatorname{Spec}(A_i)$ and each A_i is a unique factorization domain. Show that $H^1_{\mathcal{U}}(X, \mathcal{O}^*) \cong \operatorname{Pic}(X)$.