Problem Set Nine: Due Thursday, April 11

See the course website for policy on collaboration.

Textbook problems:

18.5.B A nice concrete application of Serre duality. You'll want to prove Problem 1 on this worksheet first.

18.6.G, 18.6.H, 18.6.J These are all excellent practical problems about degree. I think they should all be short.

Problem 1. An important consequence of Serre duality. Let k be a field and let $X \subset \mathbb{P}^r_k$ be a smooth, closed, geometrically irreducible subvariety of dimension d. Let $\mathcal{E}$ be a locally free sheaf on X. Show that, for $0 \leq q \leq d-1$ and N sufficiently large, we have $H^q(X, \mathcal{E} \otimes \mathcal{O}(-N)) = 0$. (Remark: We showed this without Serre duality in the case $X = \mathbb{P}^1$, do you remember how?)

Problem 2. Let k be a field and let X be an integral (Definition 5.2.4) closed subscheme of $\mathbb{P}^n_k$. Show that $H^0(X, \mathcal{O}_X)$ is a field and $[H^0(X, \mathcal{O}_X) : k] < \infty$. Hint: This comes down to an algebraic fact, which you should prove: If a finite dimensional k-algebra is an integral domain, it is a field.

Problem 3. Let f and g be relatively prime homogeneous polynomials in $k[x_0, x_1, x_2, x_3]$ of degrees a and b. Let Γ be the closed subscheme of $\mathbb{P}^3_k$ given by $f = g = 0$. If Γ is regular and irreducible, then what is its genus?