PROBLEM SET 1
DUE TUESDAY, JANUARY 18

0 Which of the following have you seen before: smooth manifolds, differential forms, analytic functions, residues (of meromorphic functions), (p, q)-forms, line bundles, vector bundles, cohomology (of topological spaces), sheaves, schemes, exact sequences, cohomology (of complexes of abelian groups), the snake lemma, derived functors?

Differential Forms

1 Let $f(x, y)$ be any smooth function from $\mathbb{R}^2 \to \mathbb{R}$. Show that there is a 1-form $\omega = g(x, y)dx + h(x, y)dy$ such that $d\omega = f(x, y)dxdy$.

2 Let ϕ be a smooth map $\mathbb{R}^n \to \mathbb{R}^n$. What is $\phi^*(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n)$?

3 On $\mathbb{R}^2 \setminus (0, 0)$, let ω be the differential form $(xdy - ydx)/(x^2 + y^2)$.

(1) Show that $d\omega = 0$.
(2) Let p be the map $t \mapsto (\cos t, \sin t)$ from $[0, 1] \to \mathbb{R}^2 \setminus (0, 0)$. What is $p^*\omega$?
(3) Let q be the map $t \mapsto (t + 1, t + 1)$ from $[0, 1] \to \mathbb{R}^2 \setminus (0, 0)$. What is $q^*\omega$?
(4) Let γ be any arc in $\mathbb{R}^2 \setminus (0, 0)$. Give (with proof) a simple description of $\int_\gamma \omega$.

Sheaves

4 Which of the following presheaves are sheaves? For those which are not sheaves, is the failure that the elements of $\mathcal{E}(U)$ we want do not exist, or that they are not unique?

(1) On X a smooth manifold, the presheaf where $\mathcal{E}(U)$ is the C^∞ functions on U?
(2) The constant presheaf: $\mathcal{E}(U) = \mathbb{Z}$ for every U?
(3) On $X = \mathbb{C}$, the presheaf of $\mathbb{C}^*$ valued functions with continuous logarithms?
(4) For a X a smooth manifold and Z a smooth submanifold, the presheaf where $\mathcal{E}(U)$ is C^∞ functions on $U \cap Z$. (Note: The unique function $\emptyset \to \mathbb{R}$ is smooth.)
(5) Let X and Y be smooth manifolds and $f : X \to Y$ a covering map. The presheaf on X where $\mathcal{E}(U) = C^\infty(f(U))$.
(6) The presheaf of isomorphism classes of line bundles: $\mathcal{E}(U)$ is the set of isomorphism classes of real line bundles over U, and ρ^U_V is restriction of line bundles. (Do this one only if you know what line bundles are.)
(7) The presheaf of cochains: Let Δ be the k-dimensional simplex and let $[\Delta, U]$ be the set of continuous maps $\Delta \to U$. Set $\mathcal{E}(U)$ to be $\mathbb{R}$ valued functions on $[\Delta, U]$. Here ρ^U_V is restriction, noting the obvious inclusion $[\Delta, V] \subset [\Delta, U]$.

5 The point of this exercise is to verify some claims made in class. Let $\mathcal{E}$ and $\mathcal{F}$ be sheaves on X and $\phi : \mathcal{E} \to \mathcal{F}$ a map of sheaves.

(1) In class, I discussed a presheaf $\mathcal{K}er$ defined on X by $\mathcal{K}er(U) = \{e \in \mathcal{E}(U) : \phi(e) = 0\}$. For $V \subseteq U$, what should the restriction map $\mathcal{K}er(U) \to \mathcal{K}er(V)$ be?
(2) Is $\mathcal{K}er(\phi)$ a sheaf?
(3) Let the presheaf $\mathcal{C}oker$ be such that $\mathcal{C}oker(U) = \mathcal{F}(U)/\phi(\mathcal{E}(U))$. What should the restriction map $\mathcal{C}oker(U) \to \mathcal{C}oker(V)$ be? Check that it is well defined!
(4) Is $\mathcal{C}oker$ a sheaf? Why or why not?