PROBLEM SET 10
DUE APRIL 5, 2011

1. Let $X = \mathbb{C}^2/\mathbb{Z}^4$ with the standard Kähler structure. So the harmonic forms are the 16 forms which can be made by wedging together dz_1, $d\overline{z_1}$, dz_2 and $d\overline{z_2}$ in all possible ways.

(1) Find a basis for the primitive part of $H^{1,1}$.
(2) How does $*$ act on $H^{2,0}$, on the primitive part of $H^{1,1}$, on $H^{0,2}$, and on ω? In each case, your answer should be either 1 or -1.

2. Let X_1 and X_2 be compact curves of genus g_1 and g_2. Then we have a Kummer-like formula $H^{p,q}(X \times Y) = \bigoplus_{p_1+p_2=p} \bigoplus_{q_1+q_2=q} H^{p_1,q_1}(X_1) \otimes H^{p_2,q_2}(X_2)$.

(1) Draw the Hodge diamond for $X \times Y$.
(2) Compute the dimensions of all the groups in the Lefschetz decomposition of $H^{\bullet,\bullet}(X_1 \times X_2)$.
(3) What is the signature of the Hermitian pairing $\langle \alpha, \beta \rangle = \int_X \alpha \wedge \overline{\beta}$ on $H^2(X \times Y, \mathbb{C})$? On which spaces is it positive definite, and on which is it negative definite?

3. The point of this exercise is to show that, for X compact Kähler, the image of $H^1(X, \mathbb{Z})$ in $H^1(X, \mathcal{O})$ is a discrete lattice.

Let H be a free abelian group of rank $2b$, with generators $h_1, h_2, \ldots, h_{2b}$. Let V be the vector space $H \otimes \mathbb{C}$; note that V comes equipped with an operation of complex conjugation. Suppose that we are given a decomposition $V \cong V^{1,0} \oplus V^{0,1}$, such that $V^{0,1}$ is the complex conjugate of $V^{1,0}$. Let $h_i = (x_i, y_i)$ be the image of h_i in the direct sum decomposition.

(1) Show that $y_i = \overline{x_i}$.
(2) Show that there is no nontrivial linear relation $\sum a_i x_i = 0$ with the a_i real.

4. Let Λ be the additive subgroup of $\mathbb{C}^2$ generated by the vectors (p_1, q_1), (p_2, q_2), (p_3, q_3) and (p_4, q_4); assume that Λ is discrete. Let $X = \mathbb{C}^2/\Lambda$.

(1) Describe an explicit basis $(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6)$ for $H_2(X, \mathbb{Z})$. We will write σ_i^* for the dual basis of $H^2(X, \mathbb{Z})$).
(2) Describe a generator ω for $H^{0,2}(X)$.
(3) Compute the 6 integrals $\int_{\sigma_i} \omega$. If you made good choices, this should not be time consuming.
(4) Let $\sum a_i \sigma_i^*$ be an element of $H^2(X, \mathbb{Z})$. What is its image in $H^{0,2}(X)$, as a multiple of ω?
(5) Exhibit a choice of (p_i, q_i) so that the map $H^2(X, \mathbb{Z}) \to H^{2,0}(X)$ is injective. (This requires some number theory.)

5. Let X be a compact Kähler manifold. Suppose that we know the operator $L : H^k(X) \to H^{k+2}(X)$. The point of this exercise is to see how Λ may be recovered from L.

(1) Give a description of the primitive cohomology in terms of L, without mentioning Λ or $*$.
(2) Show that, given the operator L, there is a unique operator $\Lambda : H^{k+2}(X) \to H^k(X)$ obeying $[\Lambda, L] = (n-k)\mathrm{Id}$. (Hint: build Λ separately on primitive cohomology and on Im L.)
(3) Show that, if L takes $H^\bullet(X, \mathbb{Q})$ to $H^\bullet(X, \mathbb{Q})$, then so do Λ and $\dagger$.
(4) Let $X = \mathbb{C}/A$, for A a discrete rank two subgroup of $\mathbb{C}$, where $\mathbb{C}$ has the standard Kähler structure. Give a choice of A such that L takes $H^\bullet(X, \mathbb{Q})$ to itself but $*$ does not.