PROBLEM SET 11
DUE APRIL 12, 2011

1. (Riemann-Roch for curves) Let X be a compact complex curve and let p be a point of X. Let $\mathbb{C}_p$ be the skyscraper sheaf at p, meaning that $\mathbb{C}_p(U) = \mathbb{C}$ if $p \in U$ and 0 if $p \notin U$. Let D be any divisor on X.

(1) Show that there is a short exact sequence $0 \to \mathcal{O}(D) \to \mathcal{O}(D+p) \to \mathbb{C}_p \to 0$.
(2) Define $\chi(D) = \dim H^0(X, \mathcal{O}(D)) - \dim H^1(X, \mathcal{O}(D))$. Show that $\chi(D+p) = \chi(D) + 1$. Deduce that there is a constant k such that $\chi(D) = \deg(D) + k$.
(3) Considering $D = \emptyset$, compute k.
(4) Suppose there is a divisor K such that $\mathcal{H}^1 \cong \mathcal{O}(K)$. Compute $\deg(K)$.

2. Let X be a compact complex curve. The point of this problem is to understand the map $\mathrm{NS} : H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$ explicitly. Let D be a divisor in X, in other words, $D = \sum a_i D_i$ for some finite set of points D_i in X. Choose a triangulation of X where all the D_i are at vertices and where every triangle contains at most one D_i. For each vertex v of the triangulation, let U_v be the open neighborhood of v described in the notes for January 20. We'll compute Čech cohomology for that open cover.

(1) Describe $\mathcal{O}(D)$ as a Čech cocycle $U_{v_i} \cap U_{v_j} \mapsto f_{ij} \in \mathcal{O}^*(U_{v_i} \cap U_{v_j})$.
(2) Describe $(NS)(\mathcal{O}(D))$ as an actual Čech cocycle for $H^2(X, \mathbb{Z})$.
(3) Recall that we can integrate to get an integer $\int_X (NS)(\mathcal{O}(D))$ (see Problem 5.(2), Set 3.) What integer do we get?

3. Let X be a complex compact curve. Let U be an open subset of X which is isomorphic to a contractible subset of $\mathbb{C}$. Let p and q be points in U, and γ a path from p to q in U.

We know that $\mathcal{O}(p-q)$ is in $\mathrm{Pic}^0(X)$, and we know that we have a surjection from $H^{0,1}(X)$ to $\mathrm{Pic}^0(X)$. In the first part of this problem, we will find a $(0,1)$-form η which maps to $\mathcal{O}(p-q)$.

Choose a connected open set V, within U, and containing γ. Let λ be a smooth function on U such that, on $U \setminus V$, we have $\lambda = \log((z-p)/(z-q))$ with the branch cut chosen along γ. Let U' be an open set with $U \cup U' = C$ and $V \cap U' = \emptyset$.

(1) Show that there is a $\overline{\partial}$-closed $(0,1)$-form η which is $\overline{\partial}\lambda$ on U and is 0 on U'.
(2) Show that the Čech cocycle $U \cap U' \mapsto \log((z-p)/(z-q))$ represents the same class in $H^1(\mathcal{O})$ as does η.
(3) By the preceding, $U \cap U' \mapsto e^{\log((z-p)/(z-q))}$ is the class in $H^1(\mathcal{O}^*)$ to which η maps. Show that this co-cycle also represents $\mathcal{O}(p-q)$.

Now, we know that $H^{0,1}(X)$ is dual to $H^{1,0}(X)$, by Serre duality. Explictly, the class η gives the map $\omega \mapsto \int_X (\eta \wedge \omega)$ from $H^{1,0}(X) \to \mathbb{C}$. In the second part of this problem, we study $\int_X (\eta \wedge \omega)$.

(4) Show that $\int_X \eta \wedge \omega = \int_U \eta \wedge \omega = \int_{\partial U} \lambda\omega$.
(5) Choose a metric on X. Let $N(\gamma, \epsilon)$ be the points of X which are within ϵ of some point of γ. Let ϵ be small enough that $N(\gamma, \epsilon) \subset U$. Show that $\int_{\partial N(\gamma,\epsilon)} \lambda\omega = \int_{\partial U} \lambda\omega$.
(6) Show that
$$\lim_{\epsilon \to 0} \int_{\partial N(\gamma,\epsilon)} \lambda\omega = (2\pi i) \int_\gamma \omega.$$

Remark: More generally, let $p_1, \ldots, p_N$ and $q_1, \ldots, q_N$ be points of X, and let γ_i be a path in X from p_i to q_i. Then the map $\omega \mapsto (2\pi i) \sum \int_{\gamma_j} \omega$ is an element of $H^{1,0}(X)^*$ which represents $\mathcal{O}(\sum p_i - \sum q_i)$. As a corollary: we obtain the **Abel-Jacobi theorem**: The line bundle $\mathcal{O}(\sum p_i - \sum q_i)$ is trivial if and only if the linear function $\omega \mapsto (2\pi i) \sum \int_{\gamma_j} \omega$ is in the lattice of maps of the form $\omega \mapsto (2\pi i) \int_\sigma \omega$, for some $\sigma \in H_1(X, \mathbb{Z})$.

4. Let a closed submanifold of a polydisc, with $H^2(X, \mathbb{Z}) = 0$.

(1) Show that $H^1(X, \mathcal{O}^*)$ is trivial.
(2) Let D be a smooth hypersurface in X. Show that there is a holomorphic function on X whose zeroes are precisely D.

5. What is wrong with the following argument? Let E be the hypersurface $y^2 = x^3 - x$ in $\mathbb{C}^2$. As we've computed before, E only has cohomology in degree ≤ 1, so $H^2(E, \mathbb{Z})$ is trivial, and $H^1(E, \mathcal{O}) = 0$ as E is a submanifold of $\mathbb{C}^2$. So, by the previous problem, for any point z in E, there is a holomorphic function f on E that vanishes once at z and nowhere else.

Let X be the compactification of E constructed in Problem Set 9, Problem 5, and let ∞ be the extra point of $E \setminus X$. Since f has a single zero on E, it must have a simple pole at ∞. So $\mathcal{O}(z - \infty)$ is trivial in $\mathrm{Pic}(X)$. Since z was arbitrary, this shows that every degree 0 divisor D has $\mathcal{O}(D)$ trivial in $\mathrm{Pic}(X)$, and thus $\mathrm{Pic}^0(X)$ is trivial.