PROBLEM SET 2
DUE JANUARY 25, 2011

1. (A complex analysis lemma) Let $s_1 < s_2$ be real numbers, let A be the annulus $\{z : s_1 < |z| < s_2\}$ in $\mathbb{C}$. Let f be an analytic function on A. For every integer n, let

$$f_n = \frac{1}{2\pi i} \oint f(z) \frac{dz}{z^{n+1}}$$

where the integral is over a circle which loops around A once.

(1) For any $r \in (s_1, s_2)$, show that there is a constant C such that $|f_n| < Cr^{-n}$.
(2) Show the following equality on A, including proving that the sum on the right converges

$$f(z) = \sum_{n=-\infty}^{\infty} f_n z^n.$$

2. Let $A^\bullet$, $B^\bullet$ and $C^\bullet$ be three complexes of abelian groups. We denote all the differential maps by ∂. Let $\alpha^i : A^i \to B^i$ and $\beta^i : B^i \to C^i$ be maps such that $\alpha\partial = \partial\alpha$, $\beta\partial = \partial\beta$ and the sequence

$$0 \to A^i \xrightarrow{\alpha} B^i \xrightarrow{\beta} C^i \to 0$$

is exact.

Recall that, in class, we "defined" a map $\delta : H^{i-1}(C^\bullet) \to H^i(A^\bullet)$ as follows: Take $[z] \in H^{i-1}(C^\bullet)$ and lift it to $z \in C^{i-1}$ with $\partial z = 0$. Find $y \in B^{i-1}$ such that $\beta y = z$. Then $\beta\partial y = 0$, so we can find $x \in A^i$ with $\alpha(x) = \partial y$. Set $\delta[z] = [x]$, where $[x]$ is the class of x in $H^i(A^\bullet)$..

The point of this exercise is to check the many unchecked claims.

(1) Show that $\partial x = 0$, so that we may speak of the class of x in $H^i(A^\bullet)$.
(2) Show that the choice of a lift z for $[z]$, and the choice of a preimage y of z, do not effect the class $[x]$ in $H^i(A)$.
(3) Show that $H^i(A^\bullet) \xrightarrow{\alpha} H^i(B^\bullet) \xrightarrow{\beta} H^i(C^\bullet)$ is exact.
(4) Show that $H^{i-1}(B^\bullet) \xrightarrow{\beta} H^{i-1}(C^\bullet) \xrightarrow{\partial} H^i(A^\bullet)$ is exact.
(5) Show that $H^{i-1}(C^\bullet) \xrightarrow{\delta} H^i(A^\bullet) \xrightarrow{\alpha} H^i(B^\bullet)$ is exact.

3. The point of this problem is to motivate why we define cohomology to vanish for injective sheaves, and to give you some experience arguing categorically.

(1) Let A_1, A_2 and B be abelian groups with maps $e_i : A_i \to B$ and $f_i : B \to A_i$ satisfying the following: $f_i \circ e_i = \mathrm{Id}$, $f_{2-i} \circ e_i = 0$ and $e_1 \circ f_1 + e_2 \circ f_2 = \mathrm{Id}$. Show that $B \cong A_1 \oplus A_2$.
(2) Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{B}$ be sheaves on a topological space X, with maps of sheaves $e_i : \mathcal{A}_i \to \mathcal{B}$ and $f_i : \mathcal{B} \to \mathcal{A}_i$ satisfying the same relations as above. Show that $H^q(\mathcal{B}) = H^q(\mathcal{A}_1) \oplus H^q(\mathcal{A}_2)$. (Hint: This has nothing to do with the details of how H^q is defined, and everything to do with category theory.)
(3) Let $\mathcal{I}$ be an injective sheaf. Suppose that we have a short exact sequence

$$0 \to \mathcal{I} \xrightarrow{e_1} \mathcal{B} \xrightarrow{f_2} \mathcal{A}_2 \to 0.$$

Show that there are maps $f_1 : \mathcal{B} \to \mathcal{I}$ and $e_2 : \mathcal{A}_2 \to \mathcal{B}$ obeying the properties of the previous question. (Hint: build f_1 first, using the definition of injectivity.)

Thus, we will have $0 \to H^0(\mathcal{I}) \to H^0(\mathcal{B}) \to H^0(\mathcal{A}_2) \to 0$ exact for any injective sheaf, so we will never be forced to take $H^1(\mathcal{I}) \neq 0$ in order to build our long exact sequences.

4. Let X be a topological space. For every point $x \in X$, let $A(x)$ be an abelian group. Let $\mathcal{A}$ be the following sheaf on X: For any open set U, $\mathcal{A}(U)$ is the abelian group of all functions on U which assign to each $x \in U$ an element of the group $A(x)$. The restriction maps are the obvious restrictions.

(1) Suppose that all of the $A(x)$'s are injective in the category of abelian groups. Show that $\mathcal{A}$ is an injective sheaf.
(2) Assume the (true) lemma that every abelian group has an injection into an injective abelian group. Show that every sheaf of abelian groups has an injection into an injective sheaf.

5. Let D be the disc $|z| < 2$ in $\mathbb{C}$. Take two copies D_1 and D_2 of D; let z_i be the coordinate on D_i. Let A_i be the annulus $1/2 < |z_i| < 2$ in D_i. Glue A_1 to A_2 by gluing z_1 to $1/z_2$. The result is a sphere, which is called the Riemman sphere or $\mathbb{CP}^1$. Define the sheaf $\mathcal{O}$ on $\mathbb{CP}^1$ where $\mathcal{O}(U)$ is functions $f : U \to \mathbb{C}$ such that $f|_{D_i}$ is analytic in z_i. In a later lecture, we will show that the higher cohomology of $\mathcal{O}$ restricted to any disc or annulus is zero, assume this for this problem.

(1) Describe $H^1(\mathbb{CP}^1, \mathcal{O})$ using Cech cohomology.
(2) Show that the group you wrote down in the previous part is trivial. (Hint: Use problem 1.)

Let $q \in \mathbb{C}$ with $0 < |q| < 1$. Let A be the annulus $|q| < |z| < |q|^{-1}$. Let E be the quotient of A where we glue z to qz whenever z and qz are both in A, so E is topologically a torus. Let $\mathcal{O}$ be the sheaf on E where $\mathcal{O}(U)$ is functions $f : U \to \mathbb{C}$ such that the pullback of f to A is analytic.

(3) Describe $H^1(E, \mathcal{O})$ using Cech cohomology.
(4) Use your description to compute $H^1(E, \mathcal{O})$.

Definition: Let X be a topological space, and $\mathcal{E}$ a sheaf on A. For A an open set of X, define a sheaf $\mathcal{E}_A$ on X by $\mathcal{E}_A(U) = \mathcal{E}(U \cap A)$ for every open set U of X. We will also define $\mathcal{E}|_A$ to be the sheaf on A gotten by $\mathcal{E}|_A(U) = \mathcal{E}(U)$ for every open $U \subseteq A \subseteq X$.

6. (Maps of sheaves are local) Let X be a topological spaces, let $\mathcal{E}$ and $\mathcal{F}$ be sheaves on X, and let U_i be an open cover of X.

(1) Let f and g be maps $\mathcal{E} \to \mathcal{F}$. Show that $f = g$ if and only if, for every U_i, we have the equality $f|_{U_i} = g|_{U_i}$ of induced maps $\mathcal{E}|_{U_i} \to \mathcal{F}|_{U_i}$.
(2) For each U_i, let g_i be a map $\mathcal{E}|_{U_i} \to \mathcal{F}|_{U_i}$. Suppose that, for any i an j, $g_i|_{U_i \cap U_j} = g_j|_{U_i \cap U_j}$. Then there is a map $f : \mathcal{E} \to \mathcal{F}$ such that $f|_{U_i} = g_i$.

7. (The Mayer-Vietoris sequence) Let X be a topological space. Let $\mathcal{E}$ be a sheaf of abelian groups on X.

(1) Let $X = A \cup B$ be an open cover of X. Define a short exact sequence
$$0 \to \mathcal{E} \to \mathcal{E}_A \oplus \mathcal{E}_B \to \mathcal{E}_{A\cap B} \to 0$$
and prove that it is exact.
(2) Show that there is a long exact sequence
$$\begin{array}{ccccc} 0 & \to & H^0(\mathcal{E}) & \to & H^0(\mathcal{E}_A) \oplus H^0(\mathcal{E}_B) \to H^0(\mathcal{E}_{A\cap B}) & \to \\ & & H^1(\mathcal{E}) & \to & H^1(\mathcal{E}_A) \oplus H^1(\mathcal{E}_B) \to H^1(\mathcal{E}_{A\cap B}) & \to \\ & & H^2(\mathcal{E}) & \to & H^2(\mathcal{E}_A) \oplus H^2(\mathcal{E}_B) \to H^2(\mathcal{E}_{A\cap B}) & \to & \cdots \end{array}$$