PROBLEM SET 3
DUE FEBRUARY 1, 2011

1. Compute:

(1) $\partial |z|^2$, on $\mathbb{C}$.
(2) $\partial \mathrm{Re}(z)/\partial z$.
(3) $\overline{\partial}\left(\overline{x} y^{-1}(x\overline{x}+y\overline{y})^{-1}\right)$, on $\mathbb{C}\times\mathbb{C}^*$.

2. Let g be the function on $\mathbb{C}$ given by $g(z)=1$ if $|z|\leq 1$ and $g(z)=0$ if $|z|>1$. Find a continuous function $f:\mathbb{C}\to\mathbb{C}$ such that, off of the unit circle, we have $\partial f/\partial \bar{z} = g$. (I am not sure whether it is easier to trace through the proof from class, or to guess the answer!)

3. Prove or disprove: Let g be a smooth function $\mathbb{C}\to\mathbb{C}$. Let U be an open subset of $\mathbb{C}$ with $g|_U = 0$. Then there is a function $f:\mathbb{C}\to\mathbb{C}$ with $\partial f/\partial\bar{z} = g$ and $f|_U = 0$.

4. Prove or disprove: Let f and g be smooth functions from $\mathbb{C}\to\mathbb{C}$, and let γ be a smooth path in $\mathbb{C}$, with endpoints a and b. Then

$$\int_\gamma \frac{\partial f(z)}{\partial z} g(z)dz = f(a)g(a) - f(b)g(g) - \int_\gamma f(z)\frac{\partial g(z)}{\partial z}dz.$$

5. Let X be a smooth manifold. Choose a triangulation of X. Let C^k be the space of $\mathbb{R}$-valued functions on the k-faces of this triangulation. Let δ be the boundary map $C^k\to C^{k+1}$ and let d be the standard map $\Omega^k\to\Omega^{k+1}$. So DeRham's theorem is an isomorphism

$$\frac{\mathrm{Ker}(\delta : C^k\to C^{k+1})}{\mathrm{Im}(\delta : C^{k-1}\to C^k)} \cong \frac{\mathrm{Ker}(d:\Omega^k\to\Omega^{k+1})}{\mathrm{Im}(d:\Omega^{k-1}\to\Omega^k)}$$

The point of this problem is to analyze this isomorphism in more detail. (Hint: Do your computations in the Cech setting.)

(1) Let γ be an oriented loop in the 1-skeleton of X. Let ω be a closed 1-form on X, and let c be a 1-co-cycle representing the same cohomology class. Show that $\int_\gamma \omega = \sum_{e\in\gamma} c(e)$.
(2) Let S be an oriented compact surface in the 2-skeleton of X. Let ω be a closed 2-form on X, and let c be a 2-co-cycle representing the same cohomology class. Express $\int_S \omega$ in terms of c and S.

6. This problem does not directly related to this week's material, but it is always good to get more practice with sheaf cohomology. And, who knows? It might just be an important lemma later!

Let X be a topological space, and let

$$0\to\mathcal{F}_n\to\mathcal{F}_{n-1}\to\cdots\to\mathcal{F}_0\to\mathcal{E}\to 0$$

be an exact sequence of sheaves of abelian groups on X. Suppose that, for all q greater than 0, and all i, we have $H^q(X,\mathcal{F}_i)=0$. Note that this resolution is oriented the *opposite way* from our standard injective resolutions. Show that

(1) For $q>0$, we have $H^q(X,\mathcal{E})=0$.
(2) The following sequence is exact:

$$0\to H^0(X,\mathcal{F}_n)\to H^0(X,\mathcal{F}_{n-1})\to\cdots\to H^0(X,\mathcal{F}_0)\to H^0(X,\mathcal{E})\to 0.$$