PROBLEM SET 4
DUE FEBRUARY 8, 2011

1. Let $\mathcal{O}$ be the sheaf of holomorphic functions on $\mathbb{C}^n$. Let H be a hypersurface in $\mathbb{C}^n$, defined by $F = 0$ for some analytic function F. Let $\mathcal{O}_H$ be the sheaf where $\mathcal{O}_H(U)$ is the holomorphic functions on $U \cap H$. Construct a short exact sequence of sheaves

$$0 \to \mathcal{O} \to \mathcal{O} \to \mathcal{O}_H \to 0.$$

Prove that it is exact.

2. This is a problem from problem set 2, made easier.

Let q be a complex number with $|q| < 1$. Let U_1 be the annulus $\{z_1 : |q| < |z_1| < |q|^{-1/4}\}$ and let U_2 be the annulus $\{z_2 : |q|^{1/4} < |z_2| < |q|^{-1}\}$. Inside each annulus, we construct two smaller annuli:

$$A_1 = \{z_1 : |q| < |z_1| < |q|^{1/2}\} \qquad B_1 = \{z_1 : |q|^{1/4} < |z_1| < |q|^{-1/4}\}$$
$$B_2 = \{z_2 : |q|^{1/4} < |z_2| < |q|^{-1/4}\} \qquad A_2 = \{z_2 : |q|^{-1/2} < |z_1| < |q|^{-1}\}$$

Yes, I want A_1 to run along the *inside* of U_1, and A_2 to run along the *outside* of U_2. Glue $z \in B_1$ to $z \in B_2$; glue $z \in A_1$ to $q^{-1}z \in A_2$. Call the resulting torus E. We'll write A and B for the parts of E coming from the A_i and the B_i.

(1) Draw U_1, U_2 and E. Color the A's red and the B's blue.
(2) The Čech complex of $\mathcal{O}$, with respect to the cover $U_1 \cup U_2$ goes
$$0 \to \mathcal{O}(U_1) \oplus \mathcal{O}(U_2) \to \mathcal{O}(A) \oplus \mathcal{O}(B) \to 0.$$
Express each of these terms explicitly as "Laurent series whose coefficients decay at such-and-such rate." Explain explicitly how to write each map in terms of Laurent series.
(3) Show that $\hat{H}^0(E, \mathcal{O}, U_1 \cup U_2) \cong \hat{H}^1(E, \mathcal{O}, U_1 \cup U_2) \cong \mathbb{C}$. Hint: First check this with formal power series, then check that imposing the convergence conditions don't change anything.

3. Let U be $\mathbb{C}^2 \setminus \{(0,0)\}$. In class, we showed that $H^1(U, \mathcal{O}) \neq 0$. So there should be a $(0,1)$-form η on U such that $\overline{\partial}\eta = 0$ but η is not of the form $\overline{\partial} f = \eta$. In this problem, we construct η.

On $(\mathbb{C}^*)^2$, notice that we have the identity:

$$\frac{\bar{x}}{y(x\bar{x} + y\bar{y})} + \frac{\bar{y}}{x(x\bar{x} + y\bar{y})} = \frac{1}{xy}. \quad (*)$$

Let

$$\eta = \overline{\partial} \frac{\bar{x}}{y(x\bar{x} + y\bar{y})} = -\overline{\partial} \frac{\bar{y}}{x(x\bar{x} + y\bar{y})}.$$

We define η on U by using the first formula on $\mathbb{C} \times \mathbb{C}^*$ and the second on $\mathbb{C}^* \times \mathbb{C}$.

(1) Show that $\overline{\partial}\eta = 0$.
We now show that η is not $\overline{\partial}$-exact. Suppose, for the sake of contradiction, that there is a function f on U such that $\overline{\partial} f = \eta$. Define a function g_1 on $\mathbb{C} \times \mathbb{C}^*$ by
$$g_1 = \frac{\bar{x}}{y(x\bar{x} + y\bar{y})} - f.$$

Define g_2 on $\mathbb{C}^* \times \mathbb{C}$ by

$$g_2 = -\frac{\bar{y}}{x(x\bar{x} + y\bar{y})} - f.$$

(2) Show that g_1 and g_2 are holomorphic.
(3) Show that $g_1 - g_2 = 1/(xy)$ and obtain a contradiction.

Remark: This is the Čech proof, made explicit. The only creative part is finding the identity $(*)$.

4. Let $U \subsetneq V$ be connected open subsets of $\mathbb{C}^2$. Suppose that we have the following amazing property: For any analytic function f on U, the function f has a holomorphic extension to V. The point of this problem is to show that, in this case $H^1(U, \mathcal{O}) \neq 0$.

Let x and y be coordinates on $\mathbb{C}^2$, and let $(a, b) \in V \setminus U$. Define a complex of sheaves on U by

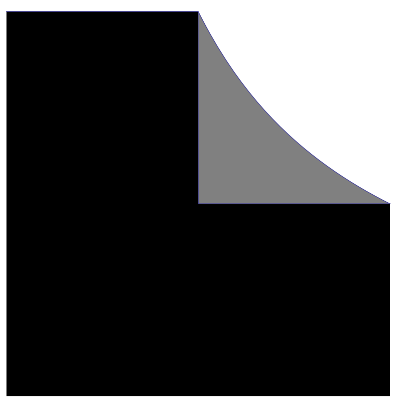
$$0 \to \mathcal{O} \xrightarrow{\begin{pmatrix} x-a \\ y-b \end{pmatrix}} \mathcal{O}^{\oplus 2} \xrightarrow{\begin{pmatrix} -(y-b) & x-a \end{pmatrix}} \mathcal{O} \to 0.$$

Here the first map means that we send f to $((x-a)f, (y-b)f)$ and the second map means that we send (g, h) to $-(y-b)g + (x-a)h$.

(1) Check that this is a complex, meaning that the composite of the nontrivial maps is zero.
(2) Show that this complex is exact, as a complex of sheaves on U.
(3) Write down the corresponding long exact sequence, and show that $H^0(U, \mathcal{O}^{\oplus 2}) \to H^0(U, \mathcal{O})$ is *not* surjective. Deduce that $H^1(U, \mathcal{O}) \neq 0$.

5. Define two open sets in $\mathbb{C}^2$ by

$$U = \{(x, y) : \min(|x|, |y|) < 1, \ \max(|x|, |y|) < 2\}$$
$$V = \{(x, y) : |x| \cdot |y| < 2, \ \max(|x|, |y|) < 2\}.$$



So U is when $(|x|, |y|)$ is in the black region, and V is when $(|x|, |y|)$ is in the union of the black and the gray regions.

(1) Let $f = \sum f_{ij} x^i y^j$ be a holomorphic function on U. Show that, for any $\epsilon > 0$, there is a constant C such that $|f_{ij}| < C(2-\epsilon)^{-\max(i,j)}$.
(2) Show that any holomorphic function f on U has a holomorphic extension to V. (Hint: Write $f = \sum f_{ij} x^i y^j$ and show that the power series converges on V.)
(3) Show that U is contractible.

So, combining the last two problems, topology is not what makes $H^1(\mathcal{O})$ vanish.