PROBLEM SET 5
DUE FEBRUARY 15, 2011

1. (Runge's approximation theorem in one variable). The point of this problem is to prove the following result: Let K be a connected, simply connected, compact, subset of $\mathbb{C}$. Let U be an open set containing K and let f be a holomorphic function defined on U. Let $\epsilon > 0$. Then there is a polynomial p with $|f(z) - p(z)| < \epsilon$ for $z \in K$.

Let γ be a closed loop in $U \setminus K$, encircling K. Recall the identity $f(z) = \frac{1}{2\pi i} \oint_{\zeta \in \gamma} f(\zeta) \frac{d\zeta}{z - \zeta}$.

(1) Show that, for any $\epsilon_1 > 0$, there is a finite collection of elements $\zeta_1, \zeta_2 \ldots \zeta_N$ in $U \setminus K$, and constants $f_1, f_2, \ldots, f_N$, such that

$$\left| f(z) - \sum \frac{f_i}{z - \zeta_i} \right| < \epsilon_1$$

for $z \in K$.

(2) Let a and b be in $\mathbb{C} \setminus K$ and suppose that, for any $z \in K$, we have $|a - b| < |z - b|$. For any positive integer k, and any $\epsilon_2 > 0$, show that there are finitely many constants $g_1, g_2, \ldots, g_N$ such that

$$\left| \frac{1}{(z-a)^k} - \sum \frac{g_i}{(z-b)^i} \right| < \epsilon_2$$

(3) Let b be in $\mathbb{C} \setminus K$ and suppose that, for any $z \in K$, we have $|z| < |b|$. For any positive integer k, and any $\epsilon_3 > 0$, show that there are finitely many constants $h_1, h_2, \ldots, h_N$ such that

$$\left| \frac{1}{(z-b)^k} - \sum h_i z^i \right| < \epsilon_3$$

(4) Prove the theorem. Note: There is still some nontrivial work to do here.

2. An example of a noncoherent sheaf: Let $\mathcal{E}$ be the following sheaf on $\mathbb{C}$: For any open set U, if $U \ni 0$, then $\mathcal{E}(U)$ is the set of holomorphic functions on U which are zero on a neighborhood of 0. If $U \not\ni 0$, then $\mathcal{E}(U)$ is the set of holomorphic functions on U.

(1) Show that the stalk $\mathcal{E}_z$ is nonzero, for $z \neq 0$, but that there is no element in $\mathcal{E}(\mathbb{C})$ whose image in $\mathcal{E}_z$ is nonzero. In other words, $\mathcal{E}$ violates the conclusion of Cartan's Theorem A.

(2) Let $\mathcal{F}$ be the cokernel of the obvious inclusion $\mathcal{E} \to \mathcal{O}$. Compute $\mathcal{F}(\mathbb{C})$. (Hint: First, what are the stalks of $\mathcal{F}$? Second, what is the definition of a cokernel?)

(3) Use the short exact sequence $0 \to \mathcal{E} \to \mathcal{O} \to \mathcal{F} \to 0$ to compute $H^1(\mathbb{C}, \mathcal{E})$. Your answer should be nonzero, showing that $\mathcal{E}$ violates the conclusion of Cartan's Theorem B.

3. Let $p(x)$ be a polynomial of degree $2g + 1$, without repeated roots. Let W be the subvariety of $\mathbb{C}^2$ where $y^2 = p(x)$.

(1) Show that every holomorphic function on W is the restriction of a holomorphic function from $\mathbb{C}^2$. (Use a resolution of $\mathcal{O}_W$.)

(2) Show that every holomorphic function on W is of the form $f(x) + g(x)y$, for f and g holomorphic functions on $\mathbb{C}$. (Hint: Start with a holomorphic function on $\mathbb{C}^2$ and replace each y^{2k} with a $p(x)^k$, justifying that all your manipulations of formal sums are legitimate.)

4. Let W be as in the previous problem. Define a holomorphic $(1,0)$-form ω on W by

$$\omega = \begin{cases} \frac{dx}{2y} & y \neq 0 \\ \frac{dy}{p'(x)} & p'(x) \neq 0 \end{cases}$$

(1) Check that, at every point on W, either $y \neq 0$ or $p'(x) \neq 0$.
(2) Check that, when both $y \neq 0$ and $p'(x) \neq 0$, the two definitions of ω agree.
(3) Let η be any holomorphic $(1,0)$ form on W. Check that $\eta = h\omega$ for some holomorphic function h on W.