PROBLEM SET 6
DUE FEBRUARY 24, 2011 – NOTE UNUSUAL DATE

1. Let L be a trivial complex line bundle on X, some real manifold. Let ∇ be a connection on L. If we choose an isomorphism between L and the product line bundle $\mathbb{C} \times X$, then sections of L can be identified with functions $X \to \mathbb{C}$. We'll write $\alpha(s)$ for the function corresponding to s, we will also write α for the identification of sections of $L \otimes \Omega^1$ with 1-forms.

We showed in class that there is a one form ω such that

$$\alpha(\nabla(s)) = d\alpha(s) + \alpha(s)\omega.$$

(1) In terms of ω, what is the map $\nabla^2 : C^\infty \otimes L \to \Omega^2 \otimes L$? When is ∇ integrable?
(2) Suppose we choose a different trivialization β of L, such that $\beta(s) = g\alpha(s)$, where g is some function $X \to \mathbb{C}^\times$. In the new coordinates, let $\beta(\nabla(s)) = d\beta(s) + \beta(s)\eta$. What is the relation between ω, η and g?

2. Let M be a connected smooth manifold and V a smooth $\mathbb{R}$ vector bundle over M. Suppose that, for each fiber V_x, we have an inner product $\langle\ ,\ \rangle$ on V_x. Let ∇ be a connection on V. Suppose that, for any two sections σ, τ of V, and any vector field X, we have the equality[1]

$$X\langle \sigma, \tau\rangle = \langle \nabla_X \sigma, \tau\rangle + \langle \sigma, \nabla_X \tau\rangle.$$

Let σ be a section of V which is ∇-constant, meaning that $\nabla(\sigma) = 0$. Show that $\langle \sigma, \sigma\rangle$ is constant.

3. Let M be a connected smooth manifold and V a smooth $\mathbb{R}$ vector bundle over M. Suppose that, for each fiber V_x, we have a linear endomorphism $E : V_x \to V_x$. Let ∇ be a connection on V. Suppose that, for any section σ of V, and any vector field X, we have the equality[2]

$$\nabla_X(E\sigma) = E\nabla_X(\sigma).$$

Let σ be a section of V which is ∇-constant, meaning that $\nabla(\sigma) = 0$. Show that $E\sigma$ is also ∇-constant.

4. This is a continuation of problems 3 and 4 from the previous problem set. Recall that p is a polynomial of degree $2g + 1$ without repeated roots, and W is the hypersurface $y^2 = p(x)$ in $\mathbb{C}^2$. In that problem, we found a holomorphic $(1, 0)$-form ω on W, given by $\omega = dx/(2y) = dy/p'(x)$. The holomorphic $(1, 0)$-forms on W are of the form $f\omega$ for some holomorphic f.

(1) Let $g(x)$ be a holomorphic function on $\mathbb{C}$. Express dg as multiple of ω.
(2) For any entire function $u(x)$, show that $u(x)y\omega$ is of the form dg for some $g(x)$.
(3) Let $h(x)$ be a holomorphic function on $\mathbb{C}$. Express $d(hy)$ as a multiple of ω.
(4) Let B be the vector space of polynomials $v(x)$ such that there is a polynomial $h(x)$ with $d(h(x)y) = v(x)\omega$. Show that $\mathbb{C}[x]/B \cong \mathbb{C}^{2g}$.
(5) **Fairly hard bonus question:** Same as the above question, with v and h entire. When I attempted this, it took some fairly messy analysis; I'm curious whether you can find a clean argument.

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[1] Most mathematicians would write $d\langle\sigma, \tau\rangle = \langle\nabla\sigma, \tau\rangle + \langle\sigma, \nabla\tau\rangle$. Exercise for those who want to work it out: Explain and justify the abuses of notation in this equation.

[2] As in the last footnote, the normal way to write this would be $\nabla(E\sigma) = E\nabla(\sigma)$. Again, what abuses of notation is this concealing?