PROBLEM SET 7
DUE MARCH 15, 2011

1. This problem takes place on $\mathbb{R}^2$, with its standard inner product.

(1) Given a 1-form $\omega = a(x,y)dx + b(x,y)dy$, what is $*\omega$?
(2) With the notation above, what is $d^*\omega$?
(3) Given a function f on $\mathbb{R}^2$, what is $\Delta_d f$?

2. Let X be a smooth compact oriented n-fold equipped with a positive definite inner product on $T_* X$.

(1) Let α be a k-form and suppose that $d^* d\alpha = 0$. Show that $d\alpha = 0$.
(2) Let β be a k-form and suppose that $\Delta\beta = 0$ and $\beta = d\alpha$. Show that $\beta = 0$. This shows that the harmonic k-forms inject into cohomology.

3. Let X be a smooth compact oriented n-fold equipped with a positive definite inner product on $T_* X$. Let $\lambda > 0$.

(1) Let ω be a k-form with $\Delta\omega = \lambda\omega$. Show that $\Delta(d\omega) = \lambda d\omega$.
(2) Let H^q_λ be the λ-eigenspace of Δ acting on $\Omega^q(X)$. Show that $0 \to H^0_\lambda \xrightarrow{d} H^1_\lambda \xrightarrow{d} \cdots \xrightarrow{d} H^n_\lambda \to 0$ is exact.

4. This problem takes place on $X = \mathbb{C}^2$, with coordinates $(z_1, z_2) = (x_1 + iy_1, x_2 + iy_2)$. Consider the inner product

$$\begin{pmatrix} g(dx_1,dx_1) & g(dx_1,dy_1) & g(dx_1,dx_2) & g(dx_1,dy_2) \\ g(dy_1,dx_1) & g(dy_1,dy_1) & g(dy_1,dx_2) & g(dy_1,dy_2) \\ g(dx_2,dx_1) & g(dx_2,dy_1) & g(dx_2,dx_2) & g(dx_2,dy_2) \\ g(dy_2,dx_1) & g(dy_2,dy_1) & g(dy_2,dx_2) & g(dy_2,dy_2) \end{pmatrix} = \begin{pmatrix} e^{x_2} & & & \\ & e^{x_2} & & \\ & & e^{x_1} & \\ & & & e^{x_1} \end{pmatrix}$$

on $T^* X$. Remark: I recommend **not** switching to dz_i and $d\overline{z}_i$ in this problem. Let $\omega = dx_1$.

(1) Check that $d\omega$, $\partial\omega$ and $\overline{\partial}\omega$ are all 0.
(2) Compute $*\omega$.
(3) Compute $d * \omega$, $\partial * \omega$ and $\overline{\partial} * \omega$.
(4) Compute $* \, d * \omega$, $* \, \partial * \omega$ and $* \, \overline{\partial} * \omega$.
(5) Compute $d * d * \omega$, $\overline{\partial} * \partial * \omega$ and $\partial * \overline{\partial} * \omega$.
(6) Compute $\Delta_d\omega$, $\Delta_\partial\omega$ and $\Delta_{\overline{\partial}}\omega$. (This should be easy, given everything that comes before.)

If you have done everything right, you should get that $\Delta\omega = 2(\Delta_\partial + \Delta_{\overline{\partial}})\omega$.

5. This problem also takes place on $X = \mathbb{C}^2$, but now with the standard inner product (so dx_1, dy_1, dx_2 and dy_2 are orthonormal). Once again, I recommend **not** switching to dz_i and $d\overline{z}_i$ in this problem. Let ω be a 1-form of the form $f dx_1$ for some function f.

(1) Compute $d * d * \omega$, $*d * d\omega$, $\partial * \overline{\partial} * \omega$, $*\overline{\partial} * \partial\omega$, $\overline{\partial} * \partial * \omega$ and $*\partial * \overline{\partial}\omega$
(2) Verify that $\Delta_d\omega = 2\Delta_\partial\omega = 2\Delta_{\overline{\partial}}\omega$.

Similar computations show that, for *any* one-form ω, the equation in part (2) holds. Thus, no holomorphic change of coordinates can turn the metric in the previous problem into the standard metric. As we will learn soon, the latter metric is Kahler, and the former is not.