PROBLEM SET 8
DUE MARCH 22, 2011

Working with Hermitian forms in coordinates

1. On $\mathbb{C}^2$, let the coordinates be $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Consider the Hermitian form $p dz_1 \otimes d\overline{z_1} + (q+ir) dz_1 \otimes d\overline{z_2} + (q-ir) dz_2 \otimes d\overline{z_1} + s dz_2 \otimes d\overline{z_2}$ for real numbers p, q, r and s.

Expand this form as $g - i\omega$, and express g and ω in the x_1, x_2, y_1, y_2 coordinates. Check directly that g is symmetric and ω is antisymmetric.

Some practice using "nice" coordinates

2. Let $g - i\omega$ be a Kähler form and let $*$ be the Hodge star with respect to g. Show that $d^*\omega = 0$.

3. Let X be a compact Kähler manifold. Let $L : \Omega^k \to \Omega^{k+2}$ be the map $\eta \to \omega \wedge \eta$. Let $\Lambda = *^{-1} L *$. Show that, for $\eta \in \Omega^k$, we have $\Lambda(L\eta) - L(\Lambda\eta) = (n-k)\eta$.

An algebraic consequence of Hodge's theorem

4. Let X be a compact Kähler manifold. Let $\mathcal{H}^p$ and $\mathcal{Z}^{p+1}$ be the sheaves of holomorphic p-forms and ∂-closed holomorphic $(p+1)$-forms respectively.

(1) For every q, show that the map $H^q(X, \mathcal{H}^p) \xrightarrow{\partial} H^q(X, \mathcal{Z}^{p+1})$ is zero. (Hint: This is really simple using harmonic representatives.)
(2) For X a compact complex manifold, even without the Kähler condition, show that $H^0(X, \mathcal{H}^0) \to H^0(X, \mathcal{H}^1)$ is zero.

5. The point of this exercise is to explore the consequences of problem 4. Just using the fact that $H^q(X, \mathcal{H}^p) \xrightarrow{\partial} H^q(X, \mathcal{Z}^{p+1})$ is zero, show the following:

(1) There is a short exact sequence $0 \to H^0(X, \mathcal{H}^1) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{H}^0) \to 0$.
(2) There is a filtration $0 \subseteq F_1 \subseteq F_2 \subseteq H^2(X, \mathbb{C})$ such that $F_1 \cong H^0(X, \mathcal{H}^2)$, $F_2/F_1 \cong H^1(X, \mathcal{H}^1)$ and $H^2(X, \mathbb{C})/F_2 \cong H^2(X, \mathcal{H}^0)$.

The formal consequences of the Kähler identities

6. The point of this exercise is to study a formal algebraic model; this will correspond to the action of the various differential operators on the λ-eigenspaces of Δ for $\lambda > 0$.

Let $\lambda > 0$. Let $V^{p,q}$, for $0 \le p, q \le n$ be $(n+1)^2$ finite dimensional vector spaces with maps ∂, $\overline{\partial}$, ∂^* and $\overline{\partial}^*$ between them shifting degrees in the obvious manners. Suppose that

(1) ∂^2, $\overline{\partial}^2$, $(\partial^*)^2$ and $(\overline{\partial}^*)^2$ are all 0.
(2) $\partial\overline{\partial} + \overline{\partial}\partial$, $\partial\overline{\partial}^* + \overline{\partial}^*\partial$, $\partial^*\overline{\partial}^* + \overline{\partial}^*\partial^*$ and $\partial^*\overline{\partial} + \overline{\partial}\partial^*$ are all 0.
(3) $\partial\partial^* + \partial^*\partial = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} = \lambda \mathrm{Id}$.

Let $Z^{p,q}$ be the subspace of $V^{p,q}$ where ∂^* and $\overline{\partial}^*$ are 0.

(1) Show that $Z^{p,q}$ is isomorphic to $\partial Z^{p,q}$ and to $\overline{\partial} Z^{p,q}$.
(2) Show that $Z^{p,q}$ is isomorphic to $\partial\overline{\partial} Z^{p,q}$. Note: This requires nontrivial diagram tracing.
(3) Show that $V^{p,q} \cong Z^{p,q} \oplus \partial Z^{p-1,q} \oplus \overline{\partial} Z^{p,q-1} \oplus \partial\overline{\partial} Z^{p-1,q-1}$.