PROBLEM SET 9
DUE MARCH 29, 2011

Problem 1 is postponed for next week.

2. Let X be a Kähler manifold, with L and Λ as discussed in class.

(1) Show that $[L, \partial]$ and $[L, \overline{\partial}]$ are 0.
(2) Show that $[\Lambda, \partial^*]$ and $[\Lambda, \overline{\partial}^*]$ are 0.
(3) Show that L and Λ commute with Δ.
(4) Show that L and Λ map harmonic forms to harmonic forms.

3. Let X be a compact Kähler manifold, with L and Λ as discussed in class. Let $(\ ,\ )$ be the Hermitian inner product on $\Omega^k(X)$, given by $(\alpha, \beta) = \int_X \alpha \overline{*\beta}$. Let $\alpha \in \Omega^k(X)$ and $\beta \in \Omega^{k+2}(X)$.

Show that $(L\alpha, \beta) = (\alpha, \Lambda\beta)$.

4. Let X be a compact complex curve of genus g. Let K be the line bundle of holomorphic $(1,0)$-forms.

(1) Show that $\dim H^0(X, K) = \dim H^1(X, \mathcal{O})$.
(2) Show that $\dim H^0(X, K) + \dim H^1(X, \mathcal{O}) = 2g$.

So we have $H^0(X, K) = H^1(X, \mathcal{O}) = g$.

5.(More of our favorite hyperelliptic curve) The aim of this polynomial is to directly compute $H^0(X, K)$ and $H^1(X, \mathcal{O})$ for a compact hyperelliptic curve. By the previous problem, we know they should be g-dimensional; the goal is to check this directly.

Let $p(x) = \sum p_k x^k$ be a polynomial of degree $2g+1$ with distinct roots and with $p(0) = 0$. Let q be the polynomial $q(x) = \sum p_{2g+2-k} x^k$, so q also has degree $2g+1$ and also has $q(0) = 0$. Let $W_1 = \{(x_1, y_1) : y_1^2 = p(x_1)\}$ in $\mathbb{C}^2$ and let $W_2 = \{(x_2, y_2) : y_2^2 = q(x_2)\}$; let $W_2 = \{(x_2, y_2) : y_2^2 = q(x_2)\}$ in $\mathbb{C}^2$. Let ω_1 and ω_2 be the 1-forms dx_1/y_1 and dx_2/y_2.

Glue $W_1 \cap (\mathbb{C}^* \times \mathbb{C})$ to $W_2 \cap (\mathbb{C}^* \cap \mathbb{C})$ by gluing (x_1, y_1) to the point with $(x_2, y_2) = (x_1^{-1}, y_1 x_1^{-g-1})$. Call the resulting curve X.

(1) Check that this formula does glue $W_1 \cap (\mathbb{C}^* \times \mathbb{C})$ to $W_2 \cap (\mathbb{C}^* \cap \mathbb{C})$.
(2) Recall that we showed on problem set 5 that every holomorphic 1-form on W_i is of the form $(f_i(x_i) + y_i g(x_i))\omega_i$ for entire functions f_i and g_i. In terms of f_i and g_i, when do we have $((f_1(x_1) + y_1 g(x_1))\omega_1)\,|_{W_1 \cap W_2} = ((f_2(x_2) + y_2 g(x_2))\omega_2)\,|_{W_1 \cap W_2}$?
(3) What are the global 1-forms on X?
(4) Similarly to what we showed before, every holomorphic function on $W_1 \cap W_2$ is of the form $f(x_1) + g(x_1)y_1$ for f and g entire on $\mathbb{C}^*$. Compute $H^1(X, \mathcal{O})$ by explicitly finding the cokernel of

$$\mathcal{O}(W_1) \oplus \mathcal{O}(W_2) \to \mathcal{O}(W_1 \cap W_2).$$

Remark: There are 4 cases one could consider in the previous problem: p can have even or odd degree, and $p(0)$ can be zero or nonzero. They are all of about the same level of difficulty; I made an arbitrary choice of which one to give you. Feel free to try the others!

6. For τ in the upper half plane, let E_τ be the genus one complex curve $\mathbb{C}/\mathrm{Span}_{\mathbb{Z}}(1,\tau)$. We fix a basis (e_1, e_2) of $H_1(E_\tau, \mathbb{Z})$ so that e_1 is represented by the cycle $\mathbb{R}/\mathbb{Z}$ and e_2 is represented by the cycle $\mathbb{R}\tau/\mathbb{Z}\tau$. Let e_1^*, e_2^* be the dual basis of $H_1(E_\tau, \mathbb{Z})$.

(1) Show that dz is a closed holomorphic $(1,0)$-form on E_τ. (This is easy.)
(2) We have the Hodge decomposition $H^1(E_\tau, \mathbb{C}) \cong H^{1,0}(E_\tau) \oplus H^{0,1}(E_\tau)$. In terms of the basis e_1^*, e_2^* for $H^1(E_\tau, \mathbb{C})$, what is the class represented by dz?
(3) Similarly, in terms of the basis e_1^*, e_2^*, give a generator of $H^{0,1}(E_\tau)$ as a subspace of $H^1(E_\tau, \mathbb{C})$.
(4) Considering $H^1(E_\tau, \mathbb{C})$ as a trivial vector bundle over the upper half plane, show that $H^{1,0}(E_\tau)$ is a holomorphic subbundle, and $H^{0,1}(E_\tau)$ is not. (To put this another way, show that none of the holomorphic sections of $H^1(E_\tau, \mathbb{C})$ lie in the subbundle $H^{0,1}(E_\tau)$.)

7. Let q be a real number, $0 < q < 1$. The Hopf surface X is defined as follows: Take $\mathbb{C}^2 \setminus \{(0,0)\}$ and quotient by the symmetry $(z_1, z_2) \mapsto (qz_1, qz_2)$. This is a standard example of a surface which cannot be given a Kähler structure.

(1) Show that $X \cong S^1 \times S^3$ as a smooth manifold.

Consider the short exact sequence $0 \to \mathrm{LC}_{\mathbb{C}} \to \mathcal{O} \xrightarrow{\partial} \mathcal{Z}^1 \to 0$ on X, where $\mathcal{Z}^1$ is d-closed $(1,0)$-forms. So we have

$$0 \to H^0(X, \mathbb{C}) \to H^0(X, \mathcal{O}) \to H^0(X, \mathcal{Z}^1) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}) \to \cdots$$

(2) The aim of the next two parts are to show that $H^0(X, \mathcal{Z}^1)$ is zero. Suppose that η is a global d-closed holomorphic $(1,0)$-form on X. Let $\tilde{\eta}$ be the pullback of η to $\mathbb{C}^2 \setminus \{(0,0)\}$. Show that $\tilde{\eta}$ extends to a d-closed $(1,0)$-form on $\mathbb{C}^2$. (Hint: See the February 1 notes.)
(3) Show that any d-closed $(1,0)$ form on $\mathbb{C}$ which is invariant under dilation by q is trivial.
(4) Give an explicit example of a d-closed 1-form on X which represents a nontrivial class in $H^1(X, \mathbb{C})$. Write your 1-form in terms of dz_i and $d\bar{z}_i$.

Remark: The image of $H^1(X, \mathbb{C})$ gives us a class in $H^1(\mathcal{O})$. It turns out that this class spans $H^1(\mathcal{O})$; see "The Cohomology of Line Bundles on Hopf Manifolds", D. Mall, *Osaka Journal of Math* (1991) **28** 999–1015.