NOTES FOR APRIL 12

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1. Statement of the Kodaira embedding theorem

Last time, we considered a complex manifold X, a holomorphic line bundle L on X, and a positive definite Hermitian form $\langle\ ,\ \rangle$ on L. In this setup, we get a connection on L,

$$\nabla = D + \overline{D}$$

where $\overline{D}$ comes from the complex structure alone, and we need to use $\langle\ ,\ \rangle$ to define D. Associated to ∇ is its curvature, which is a $(1,1)$-form Θ such that

$$\nabla^2 f = \Theta f$$

for any section f. Last time, we checked that $\frac{1}{2\pi i}\Theta$ is the closed $(1,1)$-form representing the class of L in $H^1(\mathcal{Z}^1)$; or, if X is compact Kähler, in $H^{1,1}(X)$. Today we ask what happens if $\frac{1}{2\pi i}\Theta$ is the Kähler form on X. In other words, what happens when there exists a line bundle whose curvature is the Kähler form?

We'll state the answer to this question before we prove it, in case the course ends before we finish:

Theorem 1.1 (Kodaira Embedding Theorem)**.** *If X is compact Kähler and L is a line bundle on X with curvature the Kähler form ω, then X is projective; in fact, there is an embedding*

$$\phi : X \to \mathbb{P}^M$$

for some M, such that $\phi^*\omega_{\mathbb{P}^M} = D\omega$, where D is some positive integer and $\omega_{\mathbb{P}^M}$ is the Fubini-Study metric.

The following three equivalent statements give another way to think about the hypotheses of the theorem:

(1) X is compact Kähler, and comes with a line bundle with a metric whose curvature is the Kähler form.
(2) X is compact and comes with a line bundle L with some metric such that

$$\frac{1}{2\pi i}(u, Ju)$$

is positive for arbitrary nonzero tangent vectors u.
(3) X is compact Kähler and $\omega \in H^2(X, \mathbb{Z})$.

It follows from the definition of Kähler manifolds that (1) and (2) are equivalent, since the curvature of a line bundle is automatically a (1, 1)-form, so one only needs to check the positive-definiteness condition to see that X is Kähler.

We now explain how to go from a Kähler form $\omega \in H^2(X, \mathbb{Z})$ to a line bundle, together with a metric, such that $\frac{1}{2\pi i}\Theta = \omega$.

Given X, and given $\omega \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$, we know that there is some line bundle L such that the class of L in $NS(X)$ is the class $[\omega] \in H^2_{\text{dR}}(X)$. This is because the exponential exact sequence yields

$$\ldots \to H^1(\mathcal{O}^\times) \to H^2(\mathbb{Z}) \to H^2(\mathcal{O}) \to \ldots$$

and the assumption that $\omega \in H^{1,1}(X)$ means that ω is in the kernel of the map to $H^2(X, \mathcal{O})$, so there is an element of $H^1(\mathcal{O}^\times)$ which maps to ω.

By a partition of unity argument, there is some $\langle\ ,\ \rangle$ on L such that

$$\left[\frac{1}{2\pi i}\Theta\right] = [\omega]$$

in $H^2_{\mathrm{dR}}(X)$, where Θ is the curvature of $\langle\ ,\ \rangle$. This isn't quite what we want to show; it only says that

$$\frac{1}{2\pi i}\Theta = \omega + d\alpha$$

for some one-form α. To fix this, that any other metric $\langle\ ,\ \rangle'$ on L will look like $\langle\ ,\ \rangle e^{\beta}$, where β is a smooth real-valued function. On problem set 12, problem 2, we will show that we can choose β such that the curvature really does satisfy

$$\frac{1}{2\pi i}\Theta = \omega.$$

This will use that X is compact Kähler. (It's possible that the equivalence is false if X isn't compact; at any rate, this proof won't work.)

2. Statements of the vanishing theorems

We know prove two big theorems about what happens when there's a line bundle whose class is the Kähler form. Some motivation: to compute anything in sheaf cohomology, we need to know that some cohomology groups attached to *some* sheaves vanish, since basically the only computational tool we have is to write down a lot of long exact sequences. Thus it's very useful to have vanishing theorems for some specific sheaves.

Our hypotheses will be the same for both theorems: Let X be an n-dimensional compact Kähler manifold and let L a holomorphic line bundle on X with a metric whose curvature satisfies

$$\frac{1}{2\pi i}\Theta_L = \omega.$$

In this set-up, we have

Theorem 2.1 (Kodaira Vanishing). *For $p+q>n$, we have*

$$H^q(X, \mathcal{H}^p \otimes L) = 0$$

Theorem 2.2 (Serre Vanishing). *For any holomorphic vector bundle E and any $q>0$, we have*

$$H^1(E \otimes L^{\otimes N}) = 0$$

for $N \gg 0$.

***Remark* 2.3.** — It follows from the Kodaira vanishing theorem and Serre duality that

$$H^q(X, \mathcal{H}^p \otimes L^{-1}) = 0$$

for $p+q<n$.

***Remark* 2.4.** — The question of how large N needs to be to make the theorem hold is an active research question. The proof we'll give is theoretically constructive, but the general consensus is that the techniques of the proof have been pushed as far as they can for bounding N.

3. Proof of the Kodaira vanishing theorem

To prove both of these theorems, we use a result that we proved in the course of proving the Kähler identities on March 17th: if E is a holomorphic vector bundle with connection Θ_E, then $[\Lambda, \Theta_E] = i(\Delta_D - \Delta_{\overline{D}})$.

Let $p+q>n$. We're interested in $H^q(X, \mathcal{H}^p \otimes L)$. The Dolbeault sequence identifies this cohomology group with the space of $\overline{\partial}$-closed (p,q)-forms (tensored with sections of L) mod $\overline{\partial}$-exact ones, but as we are in the compact Kähler case, we can think of these as harmonic forms, i.e. as

$$\ker(\Delta_{\overline{D}} : \Omega^{p,q} \otimes L \to \Omega^{p,q} \otimes L).$$

Let $\eta \in \ker \Delta_{\overline{D}}$. On the one hand, using the inner product $(\ ,\ )$ defined in the March 17th notes, we see that

$$(\eta, \Delta_D \eta) = (D\eta, D\eta) + (D^*\eta, D^*\eta) \geq 0,$$

and of course

$$(\eta, \Delta_{\overline{D}} \eta) = (\eta, 0) = 0.$$

Thus $(\eta, (\Delta_D \eta - \Delta_{\overline{D}})\eta) \geq 0$.

On the other hand, this same expression, after dividing by 2π, is

$$\begin{aligned}(\eta, \frac{1}{2\pi i}[\Lambda, \Theta_L]\eta) &= (\eta, [\Lambda, \omega\wedge]\eta) \\ &= (\eta, [\Lambda, L]\eta) \\ &= (n-p-q)(\eta, \eta) \\ &\leq 0.\end{aligned}$$

This is only possible if all inequalities are equalities, and hence (η, η), and so η, must be 0.

4. Interlude on connections and tensor products

The proof of Serre vanishing will be similar, but first we need some remarks on how connections behave on tensor products. Let E and F be vector bundles on X with connections ∇_E and ∇_F. Then there is a canonical connection on $E \otimes F$ given by

$$\nabla_{E\otimes F}(\sigma \otimes \tau) = \nabla_E(\sigma) \otimes \tau + \sigma \otimes \nabla_F(\tau).$$

Here we are using the canonical isomorphisms

$$(E \otimes \Omega^1) \otimes F \cong E \otimes (F \otimes \Omega^1) \cong (E \otimes F) \otimes \Omega^1.$$

We need to check lots of things about our definition of the connection on $E \otimes F$: for instance, why is it well-defined mod the relations defining the tensor product, e.g., why is

$$\nabla_{E\otimes F}((f\sigma) \otimes \tau) = \nabla_{E\otimes F}(\sigma \otimes (f\tau))?$$

We also want to check that ∇ is compatible with the structures we've already defined. If E and F have metrics $\langle\ ,\ \rangle_E$ and $\langle\ ,\ \rangle_F$ and we put the tensor product of these metrics on $E \otimes F$, and if ∇_E and ∇_F preserve $\langle\ ,\ \rangle_E$ and $\langle\ ,\ \rangle_F$, then we want to check that $\nabla_{E\otimes F}$ preserves $\langle\ ,\ \rangle_{E\otimes F}$. Also, if E and F have holomorphic structures, then $E \otimes F$ gets one, so we get operators $\overline{D}_E$, $\overline{D}_F$, and $\overline{D}_{E\otimes F}$, and we want to check that

$$\overline{D}_{E\otimes F}(\sigma \otimes \tau) = \overline{D}_E(\sigma) \otimes \tau + \sigma \otimes \overline{D}_F \tau.$$

Similarly, we want to check that an analogous formula holds for D if we have both metric and holomorphic structures. All of this will be relegated to the problem set.

***Remark* 4.1.** — On any smooth manifold, we have

$$d : \Omega^p \to \Omega^{p+1}.$$

This is not a connection on Ω^p, which is something of the form

$$\nabla : \Omega^p \to \Omega^p \otimes \Omega^1.$$

Of course, we can always find such a ∇ by taking, for instance, the Levi-Civita connection on the tangent bundle. The composition $\nabla : \Omega^p \to \Omega^p \otimes \Omega^1 \to \Omega^{p+1}$ is d if and only if ∇ is torsion-free (as the Levi-Cevita connection is.)

Now, given any connection ∇, we get a map $E \otimes \Omega^p \to E \otimes \Omega^{p+1}$ (Feb. 22). If we also have a torsion-free connection on the tangent bundle, we can ask how that map relates to $\nabla_{E\otimes\Omega^p}$: $E \otimes \Omega^p \to E \otimes \Omega^p \otimes \Omega^1$. It looks like we might be off by a sign; David isn't sure.

Now, ∇_E has a curvature $\Theta_E \in \operatorname{End}(E) \otimes \Omega^{1,1}$ and similarly for ∇_F. The relationship between the curvatures of the tensor product of line bundles with the original line bundles is given by

$$\Theta_{E\otimes F} = \Theta \otimes \operatorname{Id}_F + \operatorname{Id}_E \otimes \Theta_F,$$

using the natural map $\operatorname{End}(E) \otimes \operatorname{End}(F) \to \operatorname{End}(E \otimes F)$.

5. Proof of the Serre vanishing theorem

We have an arbitrary holomorphic vector bundle E on X, and we want to show that $H^q(E \otimes L^{\otimes N})$ vanishes for $q > 0$ and $N \gg 0$. Set $E' = E \otimes K^{-1}$, where $K = \mathcal{H}^n$. Thus we're trying to show that

$$H^q(E' \otimes K \otimes L^{\otimes N}) = 0$$

for large N. Let $\Theta_{E'}$ be the curvature of E', and let N be large (we'll say how large N needs to be later). We're interested in

$$H^q(E' \otimes L^{\otimes N} \otimes \mathcal{H}^n) = \ker(\Delta_{\overline{D}} : E' \otimes \Omega^{n,q} \otimes L^{\otimes N} \to E' \otimes \Omega^{n,q} \otimes L^{\otimes N}).$$

As before, we have $D, \overline{D}$ on $E' \otimes L^{\otimes N}$. The curvature satisfies

$$\Theta_{E'\otimes L^{\otimes N}} = \Theta_{E'} \otimes \operatorname{Id}_{L^{\otimes N}} + N \operatorname{Id}_{E'} \otimes \Theta_L,$$

where we have identified the endomorphisms of L with the trivial line bundle.

Now, suppose $\eta \in \ker \Delta_{\overline{D}} : E' \otimes L^{\otimes N} \otimes \Omega^{n,q} \to E' \otimes L^{\otimes N} \otimes \Omega^{n,q}$. We will use the same proof technique as for Kodaira vanishing. The exact same argument, namely the argument that

$$(\eta, \Delta_D \eta) \geq 0$$

and $(\eta, \Delta_{\overline{D}}\eta) = 0$, implies that $(\eta, (\Delta_D - \Delta_{\overline{D}})\eta) \geq 0$. On the other hand, this same expression, after dividing by 2π, is

$$\begin{aligned}
(\eta, [\Lambda, \frac{1}{2\pi i}\Theta_{E'\otimes L^{\otimes N}}]\eta) &= (\eta, [\Lambda, \frac{1}{2\pi i}\eta\Theta_{E'}]\eta) + N(\eta, [\Lambda, \frac{1}{2\pi i}\Theta_L]\eta) \\
&= (\eta, [\Lambda, \frac{1}{2\pi i}\Theta_{E'}]\eta) + N(\eta, [\Lambda, L]\eta) \\
&= (\eta, [\Lambda, \frac{1}{2\pi i}\Theta_{E'}]\eta) + N(n - (n+q))(\eta, \eta) \\
&= (\eta, [\Lambda, \frac{1}{2\pi i}\Theta_{E'}]\eta) - Nq(\eta, \eta).
\end{aligned}$$

Clearly if $(\eta, \eta) \neq 0$, there's an N that makes this negative, but this is circular reasoning, since the cohomology group that η lives in depends on N. Thus we need to show how to choose N independently of η.

Now, $[\Lambda, \Theta_{E'}]$ is complicated, but it is just a C^∞-linear map; that is, it's just action by some matrix in $(\operatorname{End}(E' \otimes \Omega^{p,q}))$; this matrix is entirely independent of N. Thus there is a constant C such that, for any $x \in X$, and any $v \in (E' \otimes \Omega^{n,q})_x$, we have

$$|\langle v, [\Lambda, \frac{1}{2\pi i}\Theta_{E'}]v\rangle| < C\langle v, v\rangle.$$

Here we are using the compactness of X; such a bound obviously exists in any given stalk. Now take N large enough such that $C - Nq < 0$. From our computation, we see that

$$|(\eta, [\Lambda, \frac{1}{2\pi i}\Theta_{E'}]\eta)| < \int_X C\langle \eta, \eta\rangle = C(\eta, \eta)$$

so the term is less than or equal to $(C - Nq)(\eta, \eta) \leq 0$. We deduce that $\eta = 0$, as in the proof of Kodaira vanishing.

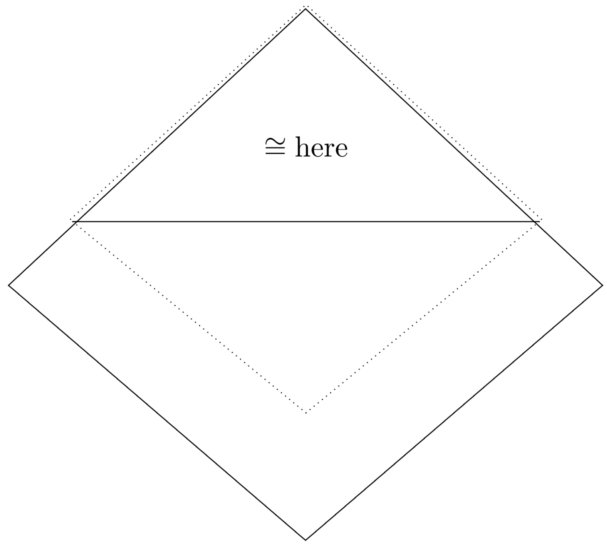

FIGURE 1. The Hodge diamond for D inside the Hodge diamond for X

6. The Lefschetz hyperplane theorem

We now give a nice topological application of the Kodaira vanishing theorem.

The Lefschetz Hyperplane Theorem Let X be compact Kähler, and L a holomorphic line bundle, such that the curvature of L is the Kähler form ω. If f is a nonzero section of L with zero locus D, and D is smooth (so $L \cong \mathcal{O}(D)$), then

$$H^{p,q}(X) \to H^{p,q}(D)$$

is an isomorphism for $p + q < n - 1$ and is injective for $p + q = n - 1$.

In other words, the Hodge diamond for D sits inside of the Hodge diamond for X as shown in the diagram, with all cohomology groups above the horizontal line equal. In particular, $H^k(X) \cong H^k(D)$ for $k < n - 1$.

The proof is simpler when $p = 0$. In this case, we use the short exact sequences of sheaves

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0$$

on X (we are collapsing the distinction between the sheaf $\mathcal{O}_D$ on D and the sheaf $\iota_*(\mathcal{O}_D)$ on X, since it doesn't affect the sheaf cohomology groups). This gives an exact sequence

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^q(X, L^{-1}) & \longrightarrow & H^q(X, \mathcal{O}) & \longrightarrow & H^q(X, \mathcal{O}_D) & \longrightarrow & H^{q+1}(X, L^{-1}) & \longrightarrow \cdots \\ & \| & & & & \| & & \| & \\ & H^{n-q}(X, K \otimes L)^\vee = 0 & & & & H^q(D, \mathcal{O}) & & H^{n-q-1}(X, K \otimes L) = 0 & \end{array}$$

The result follows from the Kodaira vanishing theorem, since both cohomology groups of $K \otimes L$ vanish.

For general p, we need to look at two short exact sequences of sheaves. Writing ι for the inclusion of D into X, we have a short exact sequence

$$0 \to \Omega^p_X \otimes \mathcal{O}(-D) \to \Omega^p_X \to \iota_*(\Omega^p_X|_D) \to 0$$

of sheaves on X. We also have a short exact sequence of sheaves on D, given by

$$0 \to \Omega^{p-1}_X|_D \otimes \mathcal{O}(-D)|_D \to \Omega^p_X|_D \to \Omega^p_D \to 0.$$

The exactness of both of these sequences can be checked in coordinates; the first map in the second sequence is described as follows: if z is a holomorphic function whose zero locus is (locally) D, we evaluate the first map on $\eta \otimes z$, by lifting η to $\widetilde{\eta}$ a $(p-1)$-form on X (defined in the same small neighborhood on which we are checking exactness); the map is

$$\eta \otimes z \mapsto \widetilde{\eta} \wedge dz,$$

and one needs to check this is well-defined independent of choice of $\widetilde{\eta}$ and of z.

Now that we have the two short exact sequences, Kodaira vanishing (in the form of Remark **??**) shows that

$$\begin{aligned} H^q(X, \Omega^p) &\cong H^q(X, \iota_*(\Omega^p_X|_D)) \\ &\cong H^q(D, \Omega^p_X|_D) \\ &\cong H^q(D, \Omega^p_D) \end{aligned}$$

for $p+q < n-1$; the same argument gives the injectivity of the desired map when $p+q = n-1$, but we can no longer deduce that the bottom map is an isomorphism, as $\dim D = n-1$.

As a consequence of the Lefschetz hyperplane theorem and the Hodge symmetry relations, we see that the cohomology of a Kähler submanifold of projective space is completely determined, except in the middle dimension.