NOTES FOR APRIL 19

GIWAN KIM

1. Zariski Toplogy

When doing algebraic algebraic geometry,

- polynomials, not holomorphic functions,
- no smooth functions, in particular no hat (bump) functions,
- don't get to use the norm $|\cdot|$ on $\mathbb{C}$.

Also work with Zariski topology on $\mathbb{C}^n$ where closed sets are of the form $\{g_1 = \cdots = g_N = 0\}$ for $g_1, \ldots, g_N \in \mathbb{C}[z_1, \ldots, z_n]$. The open sets are the complements of Zariski closed sets and this forms a topology.

Let $D(g) = \{g \neq 0\}$ for g a non-zero polynomial. The polynomial functions are rational functions of the form f/g^n for $n \in \mathbb{Z}$, $f \in \mathbb{C}[z_1, \ldots, z_n]$. Notice that $D(g) \cong \{1 - tg\} \subset \mathbb{C}^n \times \mathbb{C}$.

For any Zariski open set U, a function $f : U \longrightarrow \mathbb{C}$ is polynomial if it is a polynomial on a cover of the form $\bigcup D(g_i) = U$.

Every Zariski closed subset X of $\mathbb{C}^n$ inherits a Zariski topology ($Y \subseteq X$ is Zariski closed in X if and only if Y is Zariski closed in $\mathbb{C}^n$) and a sheaf of polynomial function Poly-$\mathcal{O}$ by restriction.

An algebraic variety is ***smooth*** if the underlying subset of $\mathbb{C}^n$ in the analytic topology is smooth or equivalently, $X \subset \mathbb{C}^n$ is smooth if for some d, X is cut out by $f_1 = \cdots = f_{n-d} = 0$ with

$$\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{n-d}}{\partial z_1} & \cdots & \frac{\partial f_{n-d}}{\partial z_n} \end{pmatrix}$$

of full rank.

2. Algebraic de Rham Complex

Algebraic de Rham complex is

$$0 \longrightarrow \text{poly-}\mathcal{O} \longrightarrow \text{poly-}\mathcal{H}^1 \longrightarrow \text{poly-}\mathcal{H}^2 \longrightarrow \cdots$$

where $\sum f_i dg_i$ with $f_i, g_i \in \mathbb{C}[z_1, \ldots, z_n]$ is a typical element of poly-$\mathcal{H}^1$. Notice that this is not exact: look at $U = \mathbb{C} \setminus \{0\}$ where $dz/z \in$ poly-$\mathcal{H}^1$ is closed but not exact. Intuitively, there are not big enough closed sets to be able to make a branch cut and log is not algebraic so is not in the sheaf.

Algebraic de Rham cohomology is $\mathbb{H}^k(0 \longrightarrow \text{poly-}\mathcal{O} \longrightarrow \text{poly-}\mathcal{H}^1 \longrightarrow \ldots)$. If $X \subset \mathbb{C}^n$ is a Zariski closed and $\mathcal{E}$ is a coherent sheaf of poly-$\mathcal{O}_X$ module then $H^q(X, \mathcal{E}) = 0$ for $q > 0$. In particular, $H^q(X, \text{any vector bundle}) = 0$ for $q > 0$. So we can compute algebraic de Rham for any open cover for which $U_{i_0} \cap \cdots \cap U_{i_p}$ is affine. If X is separated (X^{analytic} is Hausdorff) it is enough to check that each U_i is affine.

We write X^{an} when considering X as an analytic subspace of $\mathbb{C}^n$ or $\mathbb{P}^n$ and X^{Zar} when we consider X with the Zariski topology.

Theorem 1. *(Grothendieck) For any smooth quasi-projective X,*

$$H^k_{top}(X^{an}, \mathbb{C}) \cong \mathbb{H}^k(X^{Zar}, 0 \longrightarrow \textit{poly-}\mathcal{O} \longrightarrow \textit{poly-}\mathcal{H}^1 \longrightarrow \ldots)$$

where quasi-projective means Zariski open subset of Zariski closed subset of $\mathbb{P}^N$.

For X affine, above theorem combined with Cartan's Theorem says that

$$H^k_{\text{top}}(X^{\text{an}}, \mathbb{C}) \cong H^k(0 \longrightarrow \text{poly-}\mathcal{O}(X) \longrightarrow \text{poly-}\mathcal{H}^1(X) \longrightarrow \ldots).$$

3. GAGA and the projective case

For projective X, the above theorem follows from Serre's GAGA:

Theorem 2. *(Serre, GAGA) Let X be projective. There is an equivalence of categories between holomorphic coherent sheaves of X^{an} and coherent poly-$\mathcal{O}_X$-modules on X^{Zar} and compatible isomorphisms*

$$H^k(X^{Zar}, E^{Zar}) \xrightarrow{\cong} H^k(X^{an}, E^{an}).$$

Warning: Serre's morphisms are maps of $\mathcal{O}_X$-modules. In fact, the theorem is true for differential operators. See section II.6 in Deligne's "Equiations Différentielles à Points Singuiliers Règuliers", or `http://mathoverflow.net/questions/17937`.

Proof of Theorem 1 for X projective. Since X is projective, X^{an} is compact Kähler. When X is smooth projective by Serre's GAGA,

$$H^q(X^{\text{Zar}}, \text{Poly-}\mathcal{H}^p) \xrightarrow{\cong} H^q(X^{\text{an}}, \mathcal{H}^p).$$

This allows us to inductively show that $F^pH^k(X^{\text{Zar}}) \xrightarrow{\cong} F^pH^k(X^{\text{an}})$. Indeed,

$$\begin{array}{ccccccccc} \cdots \longrightarrow & F^{p+1}H^k(X^{\text{Zar}}) & \longrightarrow & F^pH^k(X^{\text{Zar}}) & \longrightarrow & H^{k-p}(X^{\text{Zar}}) & \longrightarrow \cdots \\ & \downarrow \cong \text{ by induction} & & \downarrow & & \downarrow \cong \text{ by GAGA} & \\ \cdots \longrightarrow & F^{p+1}H^k(X^{\text{an}}) & \longrightarrow & F^pH^k(X^{\text{an}}) & \longrightarrow & H^{k-p}(X^{\text{an}}) & \longrightarrow \cdots \end{array}$$

so the middle map is an isomorphism, $F^pH^k(X^{\text{Zar}}) \xrightarrow{\cong} F^pH^k(X^{\text{an}})$ by reverse induction. So we apply a theorem from April 14 to see that

$$H^k_{\text{top}}(X^{\text{an}}, \mathbb{C}) \cong F^0H^k(X^{\text{an}}) \cong F^0H^k(X^{\text{Zar}}) = H^k(0 \to \text{poly-}\mathcal{O}(X) \to \text{poly-}\mathcal{H}^1(X) \to \dots).$$

□

4. When X is not projective

Claim. Given a quasi-projective, smooth X, we can assume that $X = \overline{X} \setminus \bigcup D_i$ where $\overline{X}$ is projective and smooth and D_i are smooth divisors meeting transversally.

Proof. Since X is quasi-projective, $X = Y \setminus Z$ where Y is projective and Z is closed in Y. Resolution of singularities says that one can find $\overline{X}$ projective and smooth with a map $\overline{X} \longrightarrow Y$ which is generically 1–1 (birational). Looking at Hironaka's paper, this is what he actually proved. □

We first make a discussion back in the analytic world.

Back in the analytic world, on $\overline{X}$, we would like to talk about

$$\mathcal{MO}_X(U) = \{\text{functions meromorphic on } U \text{ and holomorphic on } X \cap U\}.$$

Define $\mathcal{O}_X(U) = \mathcal{O}(X \cap U)$ and $\mathcal{H}^p_X(U) = \mathcal{H}^p(X \cap U)$. We have a map of complexes

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{MO}_X & \longrightarrow & \mathcal{MH}^1_X & \longrightarrow \cdots \\ & \downarrow & & \downarrow & \\ 0 \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{H}^1_X & \longrightarrow \cdots \end{array}$$

and $\mathbb{H}^k(\mathcal{O}_X \to \mathcal{H}^1_X \to \dots)$ is $H^k_{\text{top}}(X, \mathbb{C})$ from previous theorem.

Claim: (Griffiths and Deligne) The above map of complexes is a quasi-isomorphism.

Proof. (sketch) We can check this locally. Without loss of generality, we may assume we are near some point with local coordinates $z_1, \dots, z_n$ where $\{z_1 = 0\}, \dots, \{z_k = 0\}$ are the D_i through this point since the D_i meet transversally.

Locally (on stalks), the complex $\mathcal{O}_X \to \mathcal{H}^1_X \to \dots$ looks like

$$0 \longrightarrow \mathbb{C}[[z_1^{\pm}, z_2^{\pm}, \dots, z_k^{\pm}, z_{k+1}, \dots, z_n]] \longrightarrow \bigoplus_i \mathbb{C}[[z_1^{\pm}, z_2^{\pm}, \dots, z_k^{\pm}, z_{k+1}, \dots, z_n]]dz_i \longrightarrow \cdots$$

since negative powers of $z_1, \dots, z_k$ are allowed on some ε-radius of $(0, 0, \dots, 0)$. The complex $\mathcal{MO}_X \to \mathcal{MH}^1_X \to \dots$ locally looks similar as above except the power series above is restricted to those with finitely many negative powers of $z_1, \dots, z_k$ convergent on some ε-neighborhood of $(0, 0, \dots, 0)$.

For example, suppose $k = n = 2$, so we are in an ball in $\mathbb{C}^2$, near which D looks like $xy = 0$. Both complexes are the sum of sequences like:

$$0 \longrightarrow \mathbb{C}x^i y^j \longrightarrow \mathbb{C}x^{i-1}y^j dx \oplus \mathbb{C}x^{i-1}y^j dx \longrightarrow \mathbb{C}x^{i-1}y^{j-1} dx \wedge dy \longrightarrow 0\,. \quad (*)$$

The difference between the two complexes are that, in $\mathcal{MO}$ we only take finitely many negative summands where as, in $\mathcal{O}$, we can have infinitely many negative summands as long as they converge rapidly enough.

The complex $(*)$ is exact except at $(i, j) = (0, 0)$, where all the maps are 0 and we get cohomology groups $(\mathbb{C}, \mathbb{C}^2, \mathbb{C})$. So we get the same cohomology groups for both convergence conditions. $\square$

We return to the proof of Theorem 1 in the case X is quasi-projective. Define the sheaf, $\mathcal{H}^p(\log D)$ on $\overline{X}$ by

$$\mathcal{H}^p(\log D)(U) = \left\{ p\text{-forms on } X \cap U \text{ which look like } \sum_{1 \leqslant i_1, \dots, i_p \leqslant n} \left(f_I(z) \bigwedge_{\substack{i \in I \\ 1 \leqslant i \leqslant k}} \frac{dz_i}{z_i} \bigwedge_{\substack{j \in I \\ k+1 \leqslant j \leqslant n}} dz_j \right) \right\}$$

where the above formula is near a point with local coordinates $z_1, \dots, z_n$ where $z_1, z_2, \dots, z_k$ cut out the D_i and f_I is holomorphic on U. This definition is independent of choice of coordinates. (Exercise!)

Then

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X(\log D) & \longrightarrow & \mathcal{H}^1(\log D) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{H}^1_X & \longrightarrow & \cdots \end{array}$$

is similarly a quasi-isomorphism so $\mathbb{H}^\bullet(\mathcal{O}_X(\log D) \to \mathcal{H}^1(\log D) \to \dots)$ computes $H^\bullet_{\text{top}}(X^{\text{an}}, \mathbb{C})$.

But note that $\Omega^p(\log D)$ is a locally-free coherent sheaf! (We described it by locally giving a basis for it as a $\mathcal{O}$-module.) So we can compute this cohomology in the Zariski topology! In short,

$$\mathbb{H}^\bullet(X^{\text{Zar}}, \text{Poly-}\mathcal{O}(\log D) \to \text{Poly-}\mathcal{H}^1(\log D) \to \dots) \quad (**)$$

also computes $H^\bullet_{\text{top}}(X^{\text{an}}, \mathbb{C})$. This is a purely algebraic formula for topological cohomology.

5. Connecting back to Theorem 1

The hypercohomology $(**)$ is often better than Theorem 1 for computations. But, for conceptual understanding, we would like to prove Theorem 1.

Define $\mathcal{MH}^p_N$ by

$$\mathcal{MH}^p_N(U) = \left\{ p\text{-forms which locally look like } \sum_I \left(f_I(z) \bigwedge_{\substack{i \in I \\ 1 \leqslant i \leqslant k}} \frac{dz_i}{z_i} \bigwedge_{\substack{j \in I \\ k+1 \leqslant j \leqslant n}} dz_j \right) (z_1 \dots z_n)^{-N} \right\}.$$

The same argument as before shows that

$$\mathbb{H}^\bullet(X^{\text{Zar}}, \text{Poly-}\mathcal{MO}_N \to \text{Poly-}\mathcal{MH}^1_N \to \dots) \cong \mathbb{H}^\bullet(X^{\text{an}}, \mathcal{MO}_N \to \mathcal{MH}^1_N \to \dots) \cong H^k_{\text{top}}(X^{\text{an}}, \mathbb{C}).$$

Observe that $\lim_{N \to \infty} \mathcal{MH}^p_N \cong \mathcal{MH}^p_X$ and $\lim_{N \to \infty} \text{Poly-}\mathcal{MH}^p_N \cong \text{Poly-}\mathcal{H}^p_X$. One can show that direct limits compute with cohomology and hypercohomology. So

$$\mathbb{H}^\bullet(X^{\text{Zar}}, \text{Poly-}\mathcal{O}_X \to \text{Poly-}\mathcal{H}^1_X \to \dots) = \varinjlim \mathbb{H}^\bullet(X^{\text{Zar}}, \text{Poly-}\mathcal{MO}_N \to \text{Poly-}\mathcal{MH}^1_N \to \dots) \cong H^k_{\text{top}}(X^{\text{an}}, \mathbb{C})$$

as was to be shown.