Notes for February 1

Scribe: Yi Su

February 4, 2011

Recap from last time: Given that K is a closed polydisc, U is an open polydisc, $K \subset U$ and ω a (p,q)-from on U with $\overline{\partial}\omega = 0$, then there is a smaller polydisc V, where $K \subset V \subset U$, and a $(p, q-1)$ form θ on V such that $\overline{\partial}\theta = \omega|_V$. This existance of local $\overline{\partial}$-antiderivative shows that at the level of sheaves, for any n-dimensional manifold, $0 \to \mathcal{H}^p \xrightarrow{\overline{\partial}} \Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \xrightarrow{\overline{\partial}} \Omega^{p,2} \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} \Omega^{p,n} \to 0$ is exact for all $q > 1$.

Because $\Omega^{p,q}$ is a C^∞ module, $H^k(\Omega^{p,q}) = 0$ for $k > 0$, by using above resolution one can get the cohomology of sheaf of holomorphic p-form

$$H^{p,q} := H^q(X, \mathcal{H}^p) = \frac{Ker(\overline{\partial} : \Omega^{p,q} \to \Omega^{p,q+1})}{Im(\overline{\partial} : \Omega^{p,q-1} \to \Omega^{p,q})} = 0 \ \forall q > 0$$

This time we are trying to strengthen the above result by showing even if we don't shrink the open polydisc U, the differential equation still can be solved.

Theorem 0.1. *Let U be an open polydisc (with possible radii ∞). Let ω be a $\overline{\partial}$-closed (p,q)-form on U. Then there exists a $(p, q-1)$ form θ on U such that $\overline{\partial}\theta = \omega$.*

Proof. As we did last time, the problem reduces to the case $p = 0$. The proof will go by induction on q. Let's do the induction step first and then do the base case when $q = 1$.

Let $K_1 \subset K_2 \subset K_3 \ldots$ be a sequence of closed polydiscs whose union is U. We want to show that given θ_i on a neighbourhood of K_i with $\overline{\partial}\theta_i = \omega|_{K_i}$, there is θ_{i+1} on a neighbourhood of K_{i+1} such that $\overline{\partial}\theta_i = \omega|_{K_{i+1}}$ and $\theta_{i+1}|_{K_i} = \theta_i$.

Let α be any $(0, q-1)$-form on a neighbourhood of K_{i+1} such that $\overline{\partial}\alpha = \omega$. Consider $\overline{\partial}(\theta_i - \alpha)$ on K_i.

$$\overline{\partial}(\theta_i - \alpha) = \overline{\partial}\theta_i - \overline{\partial}\alpha = \omega - \omega = 0$$

By induction, there is a $(0, q-2)$-form ψ with $\overline{\partial}\psi = \theta_i - \alpha$ on a neighbourhood of K_i.

Take a hat function ρ which is 1 on K_i and 0 on $U \backslash K_{i+1}$. Then $\rho\Psi$ extends the function Ψ globally. Set $\theta_{i+1} = \alpha + \overline{\partial}(\rho\Psi)$ Notice that it is defined on K_{i+1}. Then

$$\overline{\partial}\theta_{i+1} = \overline{\partial}\alpha = \omega \text{ on } K_{i+1}$$

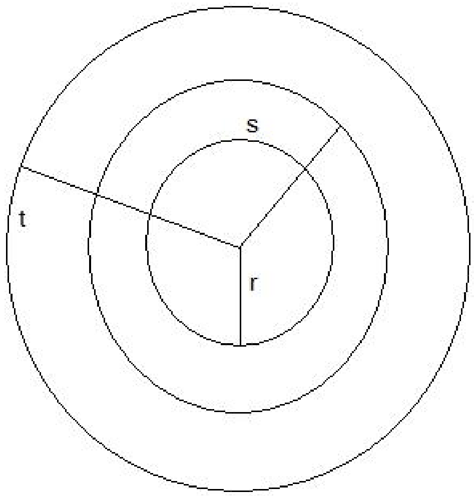
and

$$\theta_{i+1} = \alpha + \overline{\partial}(1 \cdot \Psi) = \alpha + (\theta_i - \alpha) = \theta_i \text{ on } K_i$$

This completes the induction step.

Before proving the base case, let's prove a lemma:

Lemma 0.2. *Let $B_r \subset B_s \subset B_t$ be three open polydiscs with the same center, and $r = (r_1, r_2, \ldots, r_n), s = (s_1, s_2, \ldots, s_n), t = (t_1, t_2, \ldots, t_n)$. Let f be a smooth function on B_t and holomorphic on B_s. Let $\epsilon > 0$. Then there exists a smooth function g on B_t such that $\overline{\partial} f = \overline{\partial} g$ on B_t and $|g| < \epsilon$ on B_r*



Proof. Write f as power series

$$f = \sum_{a \in \mathbb{Z}^n_{\geq 0}} f_a z_1^{a_1} z_2^{a_2} \ldots z_n^{a_n}$$

Since f is holomorphic on B_s, there is a bound on the coeffients of f. Say $|f_a| \leq M s_1^{-a_1} s_2^{-a_2} \ldots s_n^{-a_n}$, where M is a positive constant. Choose some finite subset $J \subset \mathbb{Z}^n_{\geq 0}$ such that

$$\sum_{a \in \mathbb{Z}^n_{\geq 0} \setminus J} M (r_1/s_1)^{a_1} (r_2/s_2)^{a_2} \ldots (r_n/s_n)^{a_n} < \epsilon$$

Set $g = f - \sum_{a \in J} f_a z_1^{a_1} z_2^{a_2} \ldots z_n^{a_n}$. Note that g is defined on all of B_t, because polynomials converge everywhere.Then $\overline{\partial} f = \overline{\partial} g$ on B_t, because $\overline{\partial}$ of a polynomial is zero.

$$\begin{aligned}
|g| &= |\sum_{a \in \mathbb{Z}^n_{\geq 0} \setminus J} f_a z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n}| \\
&\leq \sum_{a \in \mathbb{Z}^n_{\geq 0} \setminus J} |f_a \cdot z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n}| \\
&\leq \sum_{a \in \mathbb{Z}^n_{\geq 0} \setminus J} M (s_1)^{-a_1} (s_2)^{-a_2} \cdots (s_n)^{-a_n} \cdot r_1^{a_1} r_2^{a_2} \ldots r_n^{a_n} \\
&< \epsilon
\end{aligned}$$

□

Now let's go back to the $q = 1$ case. **This argument is significantly cleaned up from what I presented in class.**

Let $U = \cup_{i\geq 0} K_i$. Let ω be $(0,1)$-form on U. So we can find θ_i near K_i and we have $\overline{\partial}\theta_i = \omega$ near K_i. By multiplying by a hat function, we can assume all θ_i's are defined on all of U. Let $\psi_i = \theta_{i+1} - \theta_i$ near K_i. So $\overline{\partial}\psi_i = 0$ near K_i. By the above lemma, there exists ϕ_i such that $\overline{\partial}\phi_i = \overline{\partial}\psi_i$ on U and $|\phi_i| < 2^{-i}$ near K_{i-1}.

Define $\theta = \sum_{j\geq 1} \phi_j + \theta_0$ on K_i. For $j > i$. Because $\phi_j < 2^{-j}$, this sum converges uniformally and absolutely on each compact set K_i.

On K_i we have

$$\begin{aligned}
\overline{\partial}\theta &= \sum_{j=1}^{\infty} \overline{\partial}\phi_j + \overline{\partial}\theta_0 \\
&= \sum_{j=1}^{\infty} \overline{\partial}\psi_j + \overline{\partial}\theta_0 \\
&= \sum_{j=1}^{i-1} \overline{\partial}\psi_j + \overline{\partial}\theta_0 \\
&= \sum_{j=1}^{i-1} (\overline{\partial}\theta_{j+1} - \overline{\partial}\theta_j) + \overline{\partial}\theta_0 \\
&= \overline{\partial}\theta_i - \overline{\partial}\theta_0 + \overline{\partial}\theta_0 = \overline{\partial}\theta_i
\end{aligned}$$

In the second equality, because the sum is finite on K_i, we can change the $\overline{\partial}$ and the summation. This finishes the proof. $\square$

This shows $H^q(U, \mathcal{O})$ vanishes for polydiscs. A similar argument shows that it vanishes for products of discs and annuli. (We will also see another proof of this on February 3.)

The following is an example when U is not a polydisc, and we may get some non-vanishing cohomology class $H^q(U, \mathcal{O})$ for some $q > 0$, where $\mathcal{O}$ denotes the sheaf of holomorphic functions.

Let $U = \mathbb{C}^2 \backslash \{(0,0)\}$, $U_1 = \mathbb{C} \times \mathbb{C}^*$, $U_2 = \mathbb{C}^* \times \mathbb{C}$, and $U_1 \cap U_2 = \mathbb{C}^* \times \mathbb{C}^*$. Then $\{U_1, U_2\}$ is a cover of U. Each of U_1, U_2 and $U_1 \cap U_2$ is a product of discs and annuli, so:

$$H^q(U_1, \mathcal{O}) \cong H^q(U_2, \mathcal{O}) \cong H^q(U_1 \cap U_2, \mathcal{O}) = 0 \ \forall q > 0$$

Consider a complex $0 \to \mathcal{O}(U_1) \oplus \mathcal{O}(U_2) \to \mathcal{O}(U_1 \cap U_2) \to 0$. Compute the sheaf cohomology of $(\mathcal{O}, U)$ by Čech cohomology. We know that

$$\mathcal{O}(U_1) = \sum_{i\geq 0, j\in\mathbb{Z}} a_{ij} x^i y^j,\ \mathcal{O}(U_2) = \sum_{i\in\mathbb{Z}, j\geq 0} a_{ij} x^i y^j,\ \mathcal{O}(U_1 \cap U_2) = \sum_{i,j\in\mathbb{Z}} a_{ij} x^i y^j,$$

where in each case it is required that the sums converge for all $(x, y) \in (\mathbb{C}^*)^2$.

Thus,

$$H^0(\mathcal{O}, \mathbb{C}^2 \backslash \{(0,0)\}) \cong \mathbb{C}[[x, y]]$$

$$H^1(\mathcal{O}, \mathbb{C}^2 \backslash \{(0,0)\}) \cong x^{-1}y^{-1}\mathbb{C}[[x^{-1}, y^{-1}]]$$

We notice that in this case, $H^1(\mathcal{O}, \mathbb{C}^2 \backslash \{(0,0)\})$ is non-vanishing. We also see that we have proved Hartog's Theorem: Any holomorphic function on U extends to $(\mathbb{C})^2$.

Here are some clarifications and corrections from before.

1. In the computation of Čech cohomology, even if the cover is infinite, since the map from $\prod \mathcal{E}(U_{i_1} \cap U_{i_2} \cap \ldots \cap U_{i_{k-1}})$ to $\prod \mathcal{E}(U_{i_1} \cap U_{i_2} \cap \ldots \cap U_{i_k})$ only deals with finite sum each time, Čech cohomology is well-defined even for infinite cover.

2. If we have any continuous map $i : W \to X$, and $\mathcal{E}$ is a sheaf on W, for any open subset U of X, define $(i_*\mathcal{E})(U) := \mathcal{E}(i^{-1}(U))$. We have a theorem:

   Theorem 0.3. *If $i : W \to X$ is a closed inclusion, then $H^k(X, i_*\mathcal{E}) \cong H^k(W, \mathcal{E})$*

   Before we had that this is true for any inclusion. In fact it is false for open inclusions. The following is one counterexample.

   Consider the inclusion $i : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}^2$, and the sheaf of locally constant functions $\mathcal{LC}$. In this case, we know that $i_*\mathcal{LC}_{\mathbb{R}} \cong \mathcal{LC}_{\mathbb{R}}$ and the sheaf cohomology is just the usual topological cohomology. Thus,

   $$H^1(\mathbb{R}^2 \backslash \{(0,0)\}, i_*\mathcal{LC}_{\mathbb{R}}) \cong H^1(\mathbb{R}^2 \backslash \{(0,0)\}, \mathcal{LC}_{\mathbb{R}}) \cong \mathbb{R}$$

   $$H^1(\mathbb{R}^2, \mathcal{LC}_{\mathbb{R}}) \cong 0$$

   They are not equal.