# Notes for February 10

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Last class, we proved Cartan's Lemma.

**Cartan's Lemma:** Given two polyboxes $K, L$ that share an edge, open sets $U$ and $V$ around $K$ and $L$ respectively, and

$$H : U \cap V \to GL_r\mathbb{C}$$

holomorphic, then (possibly after shrinking $U, V$ to $U', V'$ still containing $K$ and $L$), we can find holomorphic

$$F : U' \to GL_r\mathbb{C}$$

$$G : V' \to GL_r\mathbb{C}$$

such that $H = F^{-1}G$.

## 1 The sheaf meaning of Cartan's lemma

Today, we will apply Cartan's lemma to study resolutions of a sheaf. Suppose we have the same setup as in Cartan's lemma, together with a sheaf $\mathcal{E}$ on $U \cup V$ and resolutions

$$0 \longrightarrow \mathcal{O}_U^{\oplus r} \xrightarrow{\cong} \mathcal{E}_U \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_V^{\oplus r} \xrightarrow{\cong} \mathcal{E}_V \longrightarrow 0$$

(where the isomorphisms denote isomorphisms of $\mathcal{O}$-modules). Then on $U \cap V$ we can compose the two isomorphisms.

$$\mathcal{O}_{U\cap V}^{\oplus r} \cong \mathcal{E}_{U\cap V} \cong \mathcal{O}_{U\cap V}^{\oplus r}$$

to get an $\mathcal{O}_{U\cap V}$-module isomorphism from $\mathcal{O}_{U\cap V}^{\oplus r}$ to itself. Let's study this isomorphism in depth. First, observe that $\mathrm{Hom}_{\mathcal{O}-\mathrm{mod}}(\mathcal{O}_{U\cap V}^{\oplus r}, \mathcal{O}_{U\cap V}^{\oplus s}) = \mathrm{Mat}_{r\times s}(\mathcal{O}_{U\cap V})$. This is noteworthy: normally, specifying a map of sheaves requires a lot of data, since you must specify the map on each open set. In this case, knowing the map on global sections determines the entire sheaf map. Indeed, basis elements of $\mathcal{O}_{U\cap V}$ restrict to basis elements of smaller open sets, whose image must be the restriction of the image of the original basis elements of $\mathcal{O}_{U\cap V}$.

In particular, the map

$$\mathcal{O}_{U\cap V}^{\oplus r} \to \mathcal{O}_{U\cap V}^{\oplus r}$$

and its inverse are both $r \times r$ matrices of holomorphic functions. Therefore, this map arises from some map $H : U \cap V \to GL_r\mathbb{C}$ with holomorphic entries. Shrinking $U$ and $V$ and using Cartan's lemma to write $H = F^{-1}G$, the following diagram commutes

$$\begin{array}{ccccccc}
\mathcal{E}_U & \longrightarrow & \mathcal{E}_{U\cap V} & = & \mathcal{E}_{U\cap V} & \longleftarrow & \mathcal{E}_V \\
\updownarrow \cong & & \updownarrow \cong & & \updownarrow \cong & & \updownarrow \cong \\
\mathcal{O}_U^{\oplus r} & \longrightarrow & \mathcal{O}_{U\cap V}^{\oplus r} & \xrightarrow{H} & \mathcal{O}_{U\cap V}^{\oplus r} & \longleftarrow & \mathcal{O}_V^{\oplus r} \\
\uparrow F & & & & & & \uparrow G \\
\mathcal{O}_U^{\oplus r} & \longrightarrow & \mathcal{O}_{U\cap V}^{\oplus r} & = & \mathcal{O}_{U\cap V}^{\oplus r} & \longleftarrow & \mathcal{O}_V^{\oplus r}
\end{array}$$

So, $(F, G)$ gives an isomorphism from $\mathcal{O}^{\oplus r} \to \mathcal{E}$ defined on $U$ and $V$ and agreeing on the overlap, thus trivializing $\mathcal{E}$ on all of $U \cup V$ (possibly after shrinking $U$ and $V$). Therefore, given a sheaf with free resolutions of length 0 on both $U$ and $V$ with the same rank, they can be glued to get a free resolution of length 0 on $U \cup V$.

## 2 Gluing resolutions

Next, we'll try to do the same for longer resolutions. First, we need some vocabulary.

**Definition:** Let $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$-modules on some complex manifold $X$. Say $\mathcal{E}$ has *global codepth* $\leq d$ if there is a resolution (an exact sequence of $\mathcal{O}_X$-modules)

$$0 \to \mathcal{O}_X^{\oplus b_d} \to \cdots \to \mathcal{O}_X^{\oplus b_0} \to \mathcal{E} \to 0$$

Say that $\mathcal{E}$ has *local codepth* $d$ if there is an open cover $\{U_i\}$ with such a resolution on each $U_i$.

**Lemma:** Let $\mathcal{E}$ be a sheaf of $\mathcal{O}_{U\cup V}$-modules on $U \cup V$, with resolutions

$$0 \to \mathcal{O}_U^{\oplus a_d} \to \cdots \to \mathcal{O}_U^{\oplus a_0} \to \mathcal{E}_U \to 0$$

$$0 \to \mathcal{O}_V^{\oplus b_d} \to \cdots \to \mathcal{O}_V^{\oplus b_0} \to \mathcal{E}_V \to 0.$$

Then, after possibly shrinking $U$ and $V$, there is a resolution

$$0 \to \mathcal{O}_{U\cup V}^{\oplus c_d} \to \cdots \to \mathcal{O}_{U\cup V}^{\oplus c_0} \to \mathcal{E}_{U\cup V} \to 0$$

(with the same $d$).

**Proof:** By induction on $d$. When $d = 0$, we have resolutions

$$0 \to \mathcal{O}_U^{\oplus a} \to \mathcal{E}_U \to 0$$

$$0 \to \mathcal{O}_V^{\oplus b} \to \mathcal{E}_V \to 0.$$

The discussion above almost completes the base case; we need only to show that $a = b$.

On $U \cap V$, we have $\mathcal{O}_{U\cap V}^{\oplus a} \cong \mathcal{O}_{U\cap V}^{\oplus b}$. Pick $z \in U \cap V$, let $\mathcal{O}_z$ be the stalk of $\mathcal{O}$ at $z$, and let $\mathcal{M} \subseteq \mathcal{O}_z$ be the ideal of holomorphic functions vanishing at $z$. Then $\mathcal{O}_z/\mathcal{M} \cong \mathbb{C}$. Taking stalks,

$$\mathcal{O}_z^{\oplus a} \cong \mathcal{O}_z^{\oplus b} \text{ as } \mathcal{O}_z\text{-modules.}$$

If we mod out by $\mathcal{M}$, then

$$(\mathcal{O}_z/\mathcal{M}\mathcal{O}_z)^{\oplus a} \cong (\mathcal{O}_z/\mathcal{M}\mathcal{O}_z)^{\oplus b} \text{ as } \mathcal{O}_z/\mathcal{M}\mathcal{O}_z\text{-modules}$$

This shows that $\mathbb{C}^a \cong \mathbb{C}^b$ as $\mathbb{C}$-vector spaces, so $a = b$ as desired.

For the inductive step suppose we have resolutions as described in the statement of the lemma. This gives the following resolutions.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathcal{O}_{U\cap V}^{\oplus a_0} & \xrightarrow{\alpha} & \mathcal{E}_{U\cap V} & \longrightarrow & 0 \\ & & & & \| & & \\ \cdots & \longrightarrow & \mathcal{O}_{U\cap V}^{\oplus b_0} & \xrightarrow{\beta} & \mathcal{E}_{U\cap V} & \longrightarrow & 0 \end{array}$$

Let $e_1, \ldots, e_{a_0}$ be basis elements of $\mathcal{O}_{U\cap V}^{\oplus a_0}$, and let $f_1, \ldots, f_{a_0}$ be preimages (under $\beta$) of $\alpha(e_1), \ldots, \alpha(e_{a_0})$. Define $\sigma : \mathcal{O}_{U\cap V}^{\oplus a_0} \to \mathcal{O}_{U\cap V}^{\oplus b_0}$ by $\sigma(e_i) = f_i$. Then $\alpha = \beta \circ \sigma$. Similarly, construct $\tau$ satisfying $\beta = \alpha \circ \tau$. Then the following diagram commutes.

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\alpha'} & \mathcal{O}_{U\cap V}^{\oplus a_0} & \xrightarrow{\alpha} & \mathcal{E}_{U\cap V} & \longrightarrow & 0 \\ & & \sigma \downarrow \uparrow \tau & & \| & & \\ \cdots & \xrightarrow{\beta'} & \mathcal{O}_{U\cap V}^{\oplus b_0} & \xrightarrow{\beta} & \mathcal{E}_{U\cap V} & \longrightarrow & 0 \end{array}$$

We would like $\sigma$ and $\tau$ to be inverses, but this dream is unrealizable if the numbers $a_0$ and $b_0$ are unequal. Our strategy will be to add factors to our resolutions to make $a_0$ and $b_0$ equal, then to modify $\sigma$ and $\tau$ so that they are inverses. Consider the following resolutions and maps.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \mathcal{O}_{U\cap V}^{\oplus a_2} & \longrightarrow & \mathcal{O}_{U\cap V}^{\oplus (a_1+b_0)} & \xrightarrow{\left(\begin{smallmatrix} \alpha' & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right)} & \mathcal{O}_{U\cap V}^{\oplus (a_0+b_0)} & \xrightarrow{(\alpha\ 0)} & \mathcal{E}_{U\cap V} & \longrightarrow & 0 \\ & & & & & & \sigma' \downarrow \uparrow \tau' & & \| & & \\ \cdots & \longrightarrow & \mathcal{O}_{U\cap V}^{\oplus b_2} & \longrightarrow & \mathcal{O}_{U\cap V}^{\oplus (a_0+b_1)} & \xrightarrow[\left(\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & \beta' \end{smallmatrix}\right)]{} & \mathcal{O}_{U\cap V}^{\oplus (a_0+b_0)} & \xrightarrow[(0\ \beta)]{} & \mathcal{E}_{U\cap V} & \longrightarrow & 0 \end{array}$$

Where $\sigma' = \left(\begin{smallmatrix} 1 & -\tau \\ \sigma & 1-\sigma\tau \end{smallmatrix}\right)$ and $\tau' = \left(\begin{smallmatrix} 1-\tau\sigma & \tau \\ -\sigma & 1 \end{smallmatrix}\right)$. First, we'll verify that this diagram still commutes. Indeed,

$$\begin{pmatrix} 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & -\tau \\ \sigma & 1-\sigma\tau \end{pmatrix} = \begin{pmatrix} \beta\sigma & \beta - \beta\sigma\tau \end{pmatrix} = \begin{pmatrix} \alpha & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha & 0 \end{pmatrix} \begin{pmatrix} 1-\tau\sigma & \tau \\ -\sigma & 1 \end{pmatrix} = \begin{pmatrix} \alpha - \alpha\tau\sigma & \alpha\tau \end{pmatrix} = \begin{pmatrix} 0 & \beta \end{pmatrix}.$$

Also observe that

$$\begin{pmatrix} 1-\tau\sigma & \tau \\ -\sigma & 1 \end{pmatrix} \begin{pmatrix} 1 & -\tau \\ \sigma & 1-\sigma\tau \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\tau \\ \sigma & 1-\sigma\tau \end{pmatrix} \begin{pmatrix} 1-\tau\sigma & \tau \\ -\sigma & 1 \end{pmatrix},$$

so $\tau'$ and $\sigma'$ are inverse maps. In particular, $\sigma'$ and $\tau'$ take values in invertible matrices.

Let $\mathcal{F}$ be $\mathcal{O}^{a_0+b_0}$ on $U$ glued by $\sigma'$ to $\mathcal{O}^{a_0+b_0}$ on $V$. By the base case, $\mathcal{F}$ is trivial over $U \cup V$, $\mathcal{F} \cong \mathcal{O}^{a_0+b_0}$ on $U \cup V$. Let $\mathcal{K} = \ker(\mathcal{O}^{a_0+b_0} \to \mathcal{E})$ on $U \cup V$. We have resolutions of $\mathcal{K}$ of length $d-1$.

$$0 \to \mathcal{O}_U^{\oplus a_d} \to \cdots \to \mathcal{O}_U^{\oplus a_2} \to \mathcal{O}_U^{\oplus(a_1+b_0)} \to \mathcal{K}_U \to 0$$

$$0 \to \mathcal{O}_V^{\oplus b_d} \to \cdots \to \mathcal{O}_V^{\oplus b_2} \to \mathcal{O}_V^{\oplus(a_0+b_1)} \to \mathcal{K}_V \to 0$$

By the inductive hypothesis, there is a resolution of $K$.

$$0 \to \mathcal{O}_{U\cup V}^{c_d} \to \cdots \to \mathcal{O}_{U\cup V}^{c_1} \to \mathcal{K}_{U\cup V} \to 0$$

So

$$\begin{array}{ccccccccccc} & & & & & & \mathcal{K} & & & & \\ & & & & & \nearrow & & \searrow & & & \\ 0 & \longrightarrow & \mathcal{O}_{U\cup V}^{\oplus c_d} & \longrightarrow \ \dots \ \longrightarrow & \mathcal{O}_{U\cup V}^{\oplus c_1} & & \longrightarrow & & \mathcal{O}_{U\cup V}^{\oplus(a_0+b_0)} & \longrightarrow & \mathcal{E}_{U\cup V} \longrightarrow 0 \end{array}$$

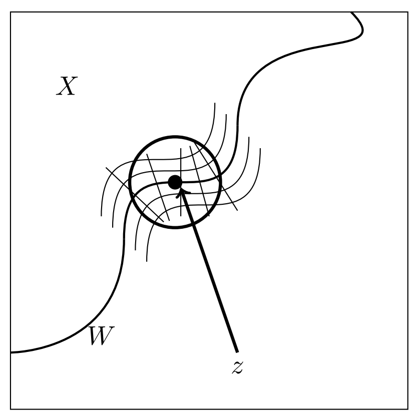
resolves $\mathcal{E}$, completing the induction. □

The moral of the lemma is that if we're given a sheaf on the union of two polyboxes that share an edge, and we're also given resolutions on these two polyboxes, then we can get a resolution on a neighborhood of the union of both boxes.

# 3 Smooth subvarieties in local equations

Next, let $W$ be a smooth, closed, complex $d$-dimensional submanifold of an $n$-dimensional complex manifold $X$, and let $z$ be a point in $W$. We want to show that on a small enough neighborhood of $z$, the sheaves $\mathcal{H}_W^p$ of holomorphic $p$-forms on $W$ have resolutions of length $n-d$ as $\mathcal{O}_X$-modules.

We can immediately reduce to the case when $X$ is an open ball in $\mathbb{C}^n$. Next, we will show that on a small enough neighborhood of $z$, we can change to a local holomorphic coordinate system $z_1, \ldots, z_n$ such that $W$ is defined by the equations $z_{d+1} = \cdots = z_n = 0$ (see picture at right).




Since $W$ is a smooth complex $d$-fold, there is some open set in $\mathbb{C}^d$, coordinates $(w_1, \ldots, w_d)$ on a neighborhood of $z \in W$, and functions $(f_1, \ldots, f_n)$ that parametrize this open set as a subset of $X \subseteq \mathbb{C}^n$.

The fact that $W$ is a *complex* submanifold means that $f_i$ are holomorphic in the $w_j$ coordinates. The fact that $W$ is a *closed* submanifold implies moreover that the map on tangent spaces is injective: the $2d \times 2n$ (real) Jacobian matrix has rank 2d. Equivalently (by complex linear algebra), the (complex) Jacobian matrix $\left(\frac{\partial f_i}{\partial w_j}\right)$ has rank $d$.

Passing to a smaller neighborhood $W'$ of $z$, we can find a submatrix

$$\begin{pmatrix} \frac{\partial f_{i_1}}{\partial w_1} & \frac{\partial f_{i_1}}{\partial w_2} & \cdots & \frac{\partial f_{i_1}}{\partial w_d} \\ \frac{\partial f_{i_2}}{\partial w_1} & \frac{\partial f_{i_2}}{\partial w_2} & \cdots & \frac{\partial f_{i_2}}{\partial w_d} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_{i_d}}{\partial w_1} & \frac{\partial f_{i_d}}{\partial w_2} & \cdots & \frac{\partial f_{i_d}}{\partial w_d} \end{pmatrix}$$

of the Jacobian that is invertible. After relabeling, we assume $(i_1, \ldots, i_d)$ is $(1, \ldots, d)$. We then modify the map $W' \to \mathbb{C}^n$ given by $(f_1, f_2, \ldots, f_d, f_{d+1}, \ldots, f_n)$ by patching on a $\mathbb{C}^{n-d}$ factor and defining $W' \times \mathbb{C}^{n-d} \to \mathbb{C}^n$ by $(f_1, f_2, \ldots, f_{d+1} + x_{d+1}, \ldots, f_n + x_n)$. This is a holomorphic map from an open set in $\mathbb{C}^n$ to an open set in $\mathbb{C}^n$ with Jacobian

$$\left(\begin{array}{c|c} \frac{\partial f_i}{\partial w_j} & \frac{\partial f_i}{\partial w_j} \\ \hline 0 & \mathrm{Id} \end{array}\right)$$

which is invertible, proving that the map is a local diffeomorphism. Therefore, by shrinking to smaller open sets $W'' \subseteq W', V \subseteq \mathbb{C}^{n-d}, X' \subseteq X \subseteq \mathbb{C}^n$, we can an inverse

$$W'' \times V \xleftarrow{(g_1, \ldots, g_n)} X'$$

with $g_i$ smooth. In fact, the $g_i$ are holomorphic. To see this, observe that the inverse is given by holomorphic functions, so the inverse map on tangent spaces is complex linear. The inverse of a complex linear map is also complex linear, so the $g_i$ must also induce complex linear maps on tangent spaces, hence the $g_i$ are holomorphic.

We've achieved our goal: we started out with a complex submanifold $W$ and a parametrization of this submanifold near a point $z$, then we padded on extra variables to get a parametrization of the ambient manifold $X$ near $z$ in such a way that the submanifold $W$ is the vanishing locus of the extra variables, then shrunk all neighborhoods until the map is a holomorphic coordinate chart.

# 4 The Köszul complex

**This section of the notes added by David Speyer. I promise that I did talk about this stuff!**

Let $W$ be as in the previous section, and $i : W \hookrightarrow X$ the inclusion. Let $\mathcal{H}^p$ be the sheaf of holomorphic $p$-forms on $W$. We want to give a resolution of $i_* \mathcal{H}^p_W$, a sheaf on $X$, by free $\mathcal{O}_X$-modules. More specifically, we want to give such a resolution locally; next time, we'll then glue the resolutions together using the earlier tools.

Pass to an open neighborhood and choose coordinates where $W$ is given by $z_{d+1} = z_{d+2} = \cdots = z_n = 0$. Then $\mathcal{H}^p_W$ is a free $\mathcal{O}_W$-module with basis $dz_{i_1} \wedge dz_{i_2} \wedge \cdots \wedge dz_{i_p}$, for $1 \leq i_1 < i_2 < \cdots < i_p \leq d$. So it is enough to give a resolution of $i_* \mathcal{O}_W$; then we can just direct sum $\binom{d}{p}$ such resolutions together.

The resolution in question is the Köszul resolution. This was already discussed as Example $2'$ on February 3. The complex looks like

$$0 \to \mathcal{O}_X \to \mathcal{O}_X^{n-d} \to \cdots \to \mathcal{O}_X^{\binom{n-d}{2}} \to \mathcal{O}_X^{n-d} \to \mathcal{O}_X \to \mathcal{O}_W \to 0.$$

Index the basis of the $r$-th term as $e(i_1 i_2 \cdots i_r)$, where $d+1 \leq i_1 < i_2 < \cdots < i_r \leq n$. Then the boundary maps are

$$\delta : e(i_1 i_2 \cdots i_r) \mapsto \sum_{j=1}^{j} (-1)^{j-1} z_{i_j} e(i_1 i_2 \cdots \widehat{i_j} \cdots i_r).$$