Notes for February 15

Scribe: Justin Campbell

Our goal for this class is to put together all of the results from the previous lectures. At this point, not much is left.

Let $X \subset \mathbb{C}^n$ be an open n-dimensional polydisc and $W \subset X$ a d-dimensional complex submanifold.

Theorem 0.1. *In this situation, $H^q(W, \mathcal{H}^p) = 0$ for all $q > 0$. Equivalently, if $i : W \to X$ is the (closed) embedding, we have $H^q(X, i_*\mathcal{H}^p_W) = 0$.*

Proof. First, we show that if K is a closed polybox in X, there is a neighborhood U of K on which $H^q(i_*\mathcal{H}^p_W) = 0$. Now for every $z \in K$, there is a small enough neighborhood N of z so that either $N \cap W = \varnothing$ or W is cut out by $n - d$ coordinates locally on N. In either case, $i_*\mathcal{H}^p_W$ has a free $\mathcal{O}_N$-module resolution of length $\leq n - d$.

Choose $\epsilon > 0$ small enough so that the box around z with side length 6ϵ is contained in N. Let B be the box of size 2ϵ centered at z, and notice that if $z' \in B$ then the box of side 2ϵ around z' is still contained in N. Thus, on such a box, there is still a resolution of length $\leq n - d$ by $\mathcal{O}$-modules.

Taking such a B for each $z \in K$, we get an open cover of K. Find a finite subcover and a uniform $\epsilon > 0$ such that for any $z' \in K$, there is such a resolution on an ϵ-box about z'. Refine K into a grid with side lengths $< \epsilon$.

Then every small box has a neighborhood with a resolution, so glue them all together, whence $i_*\mathcal{H}^q_W$ has a resolution of length $\leq n - d$ on a neighborhood U of K. Recall that we can shrink U so that $H^q(U, \mathcal{O}) = 0$ for $q > 0$. As explained before, this implies that $H^q(U, i_*\mathcal{H}^p_W) = 0$.

Here is another useful fact: consider

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{H}^p_X \longrightarrow i_*\mathcal{H}^p_W \to 0,$$

where we have named the kernel $\mathcal{E}$. As before, $\mathcal{E}$ has a finite resolution on small boxes, and there is a neighborhood of K on which $H^1(\mathcal{E}) = 0$. So there is a neighborhood of K on which $\mathcal{H}^p(X) \to \mathcal{H}^p(W)$ is actually surjective.

Now we want to show that

$$0 \longrightarrow \mathcal{H}^p(W) \longrightarrow \Omega^{p,0}(W) \longrightarrow \cdots \longrightarrow \Omega^{p,d}(W) \longrightarrow 0$$

is exact, so let $\omega \in \Omega^{p,w}(W)$ with $\overline{\partial}\omega = 0$. We know that there is a rising union of closed polydiscs $\cup_i K_i = X$ and θ_i on $W \cap K_i$ such that $\overline{\partial}\theta_i = \omega$. This is an argument we made on Feb 1 when W is itself a polydisc.

We proceed by induction on q: the case $q = 1$ is the hard part. For $q > 1$, we show that, given $K \subset K'$ and $\theta \in \Omega^{p,q-1}(K \cap W)$ with $\overline{\partial}\theta = \omega$, we can extend to $\widetilde{\theta} \in \Omega^{p,q-1}(K' \cap W)$. We know there is some θ' on K' with $\overline{\partial}\theta' = \omega$, so let $\beta = \theta - \theta'|_{K \cap W}$, whence $\overline{\partial}\beta = 0$.

So inductively, there exists α with $\overline{\partial}\alpha = \beta$, where β and α are both defined on some open neighborhood of $K \cap W$. Take a hat function ρ such that $\rho = 1$ on K and $\rho = 0$ where α is not defined. Then put $\widetilde{\theta} = \theta' - \overline{\partial}(\rho \cdot \alpha)$, so that $\overline{\partial}\widetilde{\theta} = \omega$ and $\widetilde{\theta}|_K = \theta$.

As for $q = 1$, we have $K_1 \subset K_2 \subset \cdots$ and $\theta_r \in \Omega^{p,0}(K_r \cap W)$, which we can extend to all of W using hat functions. Then put $\psi_r = \theta_r - \theta_{r-1}$, whence $\overline{\partial}\psi_r = 0$ on K_{r-1}. So we can find holomorphic σ_r on K_{r-1} such that $\psi_r = \sigma_r|_W$, and then by Runge's theorem we can shrink K_{r-1} and find a polynomial P_r such that $|\sigma_r - P_r| < 2^{-r}$ on K_{r-1}. In this way we can build

$$\theta = \theta_0 + \sum(\psi_r - P_r),$$

a function on W with $\overline{\partial} = \omega$. This finishes the proof.

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We already checked that

$$0 \longrightarrow LC_{\mathbb{C}} \longrightarrow \mathcal{H}^0 \longrightarrow \mathcal{H}^1 \longrightarrow \cdots \longrightarrow \mathcal{H}^d \longrightarrow 0$$

is an exact sequence of sheaves on W, so

$$H^p(W, LC_{\mathbb{C}}) \cong H^p_{top}(W, LC_{\mathbb{C}}) \cong \frac{\{\partial - \text{closed holomorphic } p - \text{forms}\}}{\{\partial - \text{exact holomorphic } p - \text{forms}\}}.$$

In particular, with notation as before, $H^p_{top}(W, \mathbb{C}) = 0$ for all $p > d$, even though $\dim_{\mathbb{R}} W = 2d$.