NOTES FOR FEBRUARY 17

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Three ways to think about vector bundles:

(1) as fiber bundles
(2) in terms of gluing data
(3) as locally free sheaves

1. Vector Bundles as Fiber Bundles

First, we'll introduce some notation and vocabulary:
Let $\pi : E \to X$ be a continuous map of topological spaces. Define

$$E \times_X E = \{(e_1, e_2) \in E \times E \mid \pi(e_1) = \pi(e_2)\}.$$

A $\mathbb{C}$-***vector bundle*** is the data of X, E, $\pi : E \to X$, $m : \mathbb{C} \times E \to E$, and $p : E \times_X E \to E$ such that there is an open cover U_i of X such that over U_i, the tuple

$$(U_i, \pi^{-1}(U_i), \pi|_{\pi^{-1}(U_i)}, m|_{\mathbb{C}\times\pi^{-1}(U_i)}, p|_{\pi^{-1}(U_i)\times_X\pi^{-1}(U_i)})$$

is isomorphic to

$$(U_i, U_i \times \mathbb{C}^r, \text{obvious projection, scalar multiplication, addition}).$$

We also have $\mathbb{R}$-vector bundles, defined analogously.

Example 1.1. $X \times \mathbb{R}$ is an $\mathbb{R}$-vector bundle.

Example 1.2. The Möbius Strip is an $\mathbb{R}$-vector bundle–locally it is $S^1 \times \mathbb{R}$.

Example 1.3. If X is any smooth manifold, T_*X (the tangent bundle) and T^*X (the cotangent bundle) are vector bundles. So are $\bigwedge^k T^*$, $T^* \otimes T_*$, $\mathrm{Sym}^2(T^* \oplus T_*)$, etc.
Note: $\oplus$, $\otimes$, $\bigwedge^\bullet$, $\mathrm{Sym}^\bullet$ *can all be identified locally and glued together.*

If X is a complex manifold, then $(T_*)_\mathbb{R}X$ comes with an action J (corresponding to multiplication by i) on each fiber. For $(T_*) \otimes \mathbb{C}$, J acts (functorially) and

$$T_* \otimes \mathbb{C} \cong T_{1,0} \oplus T_{0,1},$$

where $T_{1,0}$ is the i eigenspace, and $T_{0,1}$ is the $-i$ eigenspace for J. Sections of $T_{1,0}$ are $(1,0)$-vector fields, locally $\sum f_i \frac{\partial}{\partial z_i}$, where the f_i's are smooth, and sections of $T_{0,1}$ are $(0,1)$-vector fields. Similarly,

$$T^* \otimes \mathbb{C} \cong T^{1,0} \oplus T^{0,1}.$$

That is,

$$(\text{differential 1-forms with } \mathbb{C}\text{-values}) \cong ((1,0) - \text{forms}) \oplus ((0,1) - \text{forms})\,.$$

Definition 1.4. A ***map of vector bundles*** is

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & E' \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi'} \\ X & \xrightarrow{=} & X \end{array}$$

such that the above diagram commutes and $\alpha(e_1 + e_2) = \alpha(e_1) + \alpha(e_2)$ (when defined), and $\alpha(\lambda e) = \lambda\alpha(e)$, λ a scalar.

2. Vector Bundles in Terms of Gluing Data

Starting with the map $\pi : E \to X$, let U_i be an open cover of X so that we have

$$\phi_i : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{C}^r.$$

For U_i and U_j in our cover, we define the map $\psi_{j\leftarrow i}$ to be the composition

$$\psi_{j\leftarrow i} : (U_i \cap U_j) \times \mathbb{C}^r \xrightarrow{\phi_i^{-1}} \pi^{-1}(U_i \cap U_j) \xrightarrow{\phi_j} (U_i \cap U_j) \times \mathbb{C}^r.$$

Then $\psi_{j\leftarrow i}(u, v) = (u, g_{j\leftarrow i}v)$, where $g_{j\leftarrow i}$ is a continuous map from $U_i \cap U_j \to GL_r\mathbb{C}$. On the triple intersection $U_i \cap U_j \cap U_k$, we have

$$g_{k\leftarrow j} \cdot g_{j\leftarrow i} = g_{k\leftarrow i},$$

since

$$\phi_k \circ \phi_j^{-1} \circ \phi_j \circ \phi_i^{-1} = \phi_k \circ \phi_i^{-1}.$$

Note that it follows that $g_{i\leftarrow i} = \mathrm{Id}$ and $g_{i\leftarrow j} g_{j\leftarrow i} = \mathrm{Id}$.

This brings us to the following theorem.

Theorem 2.1. *Given a topological space X, an open cover U_i, and continuous maps $g_{j\leftarrow i} : U_i \cap U_j \to GL_r\mathbb{C}$, such that*

$$g_{k\leftarrow j} \cdot g_{j\leftarrow i} = g_{k\leftarrow i},$$

there is a unique (up to isomorphism) vector bundle $\pi : E \to X$ giving rise to them.

Proof (Sketch). Take $\bigsqcup(U_i \times \mathbb{C}^r)$. For $u \in U_i \cap U_j$, glue (u, v) in $U_i \times \mathbb{C}^r$ to $(u, g_{j\leftarrow i}(u)v)$ in $U_j \times \mathbb{C}^r$. Let E be the result of these gluings, etc. $\square$

A vector bundle is...

- smooth if we can arrange that the maps

$$g_{j\leftarrow i} : U_i \cap U_j \to GL_r\mathbb{C}$$

  are smooth (this makes sense for X a smooth manifold).
- holomorphic if we can arrange that the $g_{j\leftarrow i}$ are holomorphic (this makes sense for a complex manifold).
- a local system if we can arrange that the $g_{j\leftarrow i}$ are locally constant.

The following is an example of where local systems come from.

Example 2.2. Let $p : Y \to X$ be a submersion (i.e. a smooth map). Locally, this looks like $M \times U \to U$ for some M. We want to define a vector bundle that is locally $H^q(M, \mathbb{Z}) \times U \to U$. On $M \times (U_i \cap U_j) \to U_i \cap U_j$, glue by

$$g : U_i \cap U_j \to \mathrm{Aut}(M)$$

and

$$g^* : U_i \cap U_j \to GL_r(\mathbb{Z}),$$

where r is the dimension of $H^q(M, \mathbb{Z})$. g^* must be locally constant since the target is discrete! Let $R^q p_*$ be the vector bundle on X whose fibers are $H^q(p^{-1}(x), \mathbb{R})$, glued by g^*.

Now, in order to construct a map between vector bundles (over the same base X) equipped with gluing data, we need to discuss refinements of their covers.

Given gluing data U_i, $g_{j\leftarrow i}$, a ***refinement*** of the cover U_i is:

- An open cover $V_{i'}$, and
- for each i', an index i, such that $V_{i'} \subseteq U_i$.

Now, for $V_{i'}$ and $V_{j'}$, with corresponding indices i and j, so that $V_{i'} \cap V_{j'} \subset U_i \cap U_j$, let

$$h_{j\leftarrow i} = g_{j\leftarrow i}|_{V_{i'} \cap V_{j'}}.$$

This is again gluing data, and gives an isomorphic vector bundle.

Now, let X be a topological space, and let $(U_i, g_{j\leftarrow i})$ and $(V_i, h_{j\leftarrow i})$ be two sets of gluing data over X, where the rank of U_i is r and the rank of V_i is s. To give a map from the first to second vector bundle, find a common refinement W_i of U_i and V_i. Let our new maps be $g'_{j\leftarrow i}$ and $h'_{j\leftarrow i}$. On each W_i, give a map $a_i : W_i \to \mathrm{Mat}_{s\times r}(\mathbb{C})$ such that $h'_{j\leftarrow i} \cdot a_i = a_j \cdot g'_{j\leftarrow i}$ on $W_i \cap W_j$.

The two pairs (W_i, a_i) and (W_i', a_i') give the same map of vector bundles if and only if there is a refinement W_i'' where $a_i|_{W_{i'}''} = a_{i'}'|_{W_{i''}''}$.

A map of smooth vector bundles/holomorphic vector bundles/local systems means that the a_i are smooth/holomorphic/locally constant.

3. Vector Bundles as Locally Free Sheaves

Given a map $\pi : E \to X$, and $U \subset X$, let $\mathcal{E}(U) = \{\sigma : U \to \pi^{-1}(U) \mid \pi \circ \sigma = \mathrm{Id}\}$. $\mathcal{E}$ is a sheaf, with the restriction maps being the restriction of functions. This is a sheaf of complex vector spaces, with pointwise addition and scalar multiplication. If E is smooth/holomorphic/a local system, we can define the sheaf of smooth/holomorphic/locally constant sections.

Warning: Given a holomorphic vector bundle, we can talk about the sheaf of smooth sections.

Example 3.1. $\mathcal{H}^p$ is the holomorphic sections of $T^{p,0} = \bigwedge^p T^{1,0}$ while $\Omega^{p,0}$ is the smooth sections of $T^{p,0}$. In general, $T^{p,q} = \bigwedge^p T^{1,0} \otimes \bigwedge^q T^{0,1}$ and $\Omega^{p,q}$ is smooth sections of $T^{p,q}$.

Note: Not all sheaves are sections of vector bundles. Consider the closed map $W \hookrightarrow X$. Then $i_*\mathcal{O}_W$ is *not* a vector bundle.

- In the topological vector bundle setting, $\mathcal{E}$ is a continuous $\mathbb{C}$-valued functions module. Given a section $\sigma : U \to \pi^{-1}(U)$, with $f : U \to \mathbb{C}$, $f \cdot \sigma$ is also a section.
- Smooth vector bundles give C^∞-modules.
- Holomorphic vector bundles give $\mathcal{O}$-modules.
- Local systems give $\mathrm{LC}_\mathbb{C}$-modules,

A map of C^∞-modules $\mathcal{E} \to \mathcal{F}$ is equivalent to a map of smooth vector bundles $E \to F$.

Note: Consider $C^\infty \xrightarrow{d} \Omega^1$. As $d(f \cdot \sigma) \neq f d(\sigma)$, this is not a map of vector bundles; it is what is called a connection.

Theorem 3.2 (Swan - Serre)**.** *The category of smooth vector bundles/holomorphic vector bundles/local systems on X is equivalent to the category of locally free C^∞-/$\mathcal{O}$-/$LC_\mathbb{C}$-modules.*

What is locally free? It means that there is an open cover U_i such that, on U_i, $\mathcal{E} \cong C^\infty(U_i)^{\oplus r}$ / $\mathcal{O}(U_i)^{\oplus r}$ / $\mathrm{LC}_\mathbb{C}(U_i)^{\oplus r}$.

Remark 3.3. *In the C^∞ world, if we know the $C^\infty(X)$-module structure of $\mathcal{E}(X)$, this determines $\mathcal{E}$ as a sheaf of C^∞-modules.*

Proof (Sketch). Given $x \in X$, and $\pi^{-1}(x) \cong \mathcal{E}(X) \otimes_{C^\infty(X)} \mathbb{C}$, where $\mathbb{C}$ is a C^∞-module by "value at x". In other words $\sigma \equiv \sigma'$ if $\sigma - \sigma' = \sum f_i T_i$, where the f_i are functions vanishing at x and the T_i are global sections.

Locally, if $\sigma(x) = \sigma'(x)$, then $\sigma(x) - \sigma'(x) = \sum z_i T_i$, where the z_i are local coordinates.

If you have a section $\sigma \in \mathcal{E}(U)$ and $f \in C^\infty(U)$, find an open cover V_i of U such that $\bar{V}_i$ is compact in U, and a hat function J_i, which is 1 on V_i and 0 on $X \setminus U$. Then $T_i\sigma$ and $T_i f$ extend to sections in $\mathcal{E}(X)$ and $C^\infty(X)$. So on V_i, $f \cdot \sigma = (T_i f)(T_i \sigma)$. So we know what $f\sigma|_{V_i}$ is, so we know what $f\sigma$ is. ∎