NOTES FOR FEBRUARY 22

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1. Defining Connections

Motivation for connections: We'd like to compare vectors in different fibers of a vector bundle. At first, we'll talk about how to do this infinitesimally, by taking derivatives of sections. Later, we'll talk about parallel transport, which addresses this goal more directly.

Definition. *(Incomplete)*

A connection is a map which at every point $x \in X$, given a smooth section σ of E, a vector bundle, and a tangent vector v at x, outputs a vector in the fiber E_x such that

(1) $\nabla_{v_1+v_2}\sigma = \nabla_{v_1}\sigma + \nabla_{v_2}\sigma$
(2) $\nabla_{cv}\sigma = c\nabla_v\sigma$.
(3) $\nabla_v(\sigma + \tau)$
(4) $\nabla_v f\sigma(x) = v(f)(x)\sigma(x) + f(x)\nabla_v\sigma$

Here f is a C^∞ function, σ a C^∞-section, c is a scalar and $v(f)$ is the derivative of f by v.

This definition is incomplete in that we have no condition on how things vary with x.

Definition. *(Complete)*

A connection ∇ takes as input a smooth section σ, a smooth vector field V and outputs a smooth section $\nabla_v\sigma$ such that

(1) $\nabla_{V_1+V_2}\sigma = \nabla_{V_1}\sigma + \nabla_{V_2}\sigma$
(2) $\nabla_{fV}\sigma = f\nabla_V\sigma$
(3) $\nabla_V(\sigma + \tau) = \nabla_V\sigma + \nabla_V\tau$
(4) $\nabla_V f\sigma = V(f)\sigma + f\nabla\sigma$

Fundamental example: $X \times R \to X$, the trivial bundle. Let $\nabla_V(f) = V(f)$. This is the archetypical connection on the trivial bundle, but not the only one we can define.

Proposition. *$(\nabla_V\sigma)(x)$ only depends on V at x and σ in a neighborhood of x.*

Proof. If $V_1(x) = V_2(x)$, then $V_1 = V_2 + \sum f_i U_i$ where the f_i are C^∞-functions with $f_i(x) = 0$ and the U_i are C^∞ vector fields. This implies

$$\nabla_{V_1}(\sigma) = \nabla_{V_2}(\sigma) + \sum \nabla_{f_i U_i}(\sigma) = \nabla_{V_2}(\sigma) + \sum f_i \nabla_{U_i}(\sigma)$$

The second sum is 0 at x, so $\nabla_{V_1}(\sigma)(x) = \nabla_{V_2}(\sigma)(x)$ □

Similarly, if $\sigma_1 = \sigma_2$ near x, then $\sigma_1 = \sigma_2 + \sum f_i\tau_i$ for $f_i \equiv 0$ near x, τ_i smooth sections.

$$\nabla_V(\sigma_1) = \nabla_V(\sigma_2) + \sum \nabla_V(f_i\tau_i) = \nabla_V(\sigma_2) + \sum f_i\nabla_V(\tau_i) + \sum V(f_i)\tau_i = 0$$

We have some stronger statement in terms of σ:

Proposition. *If γ is a curve through x, with $\dot\gamma = V(x)$, then $\nabla_V\sigma(x)$ only depend on $\sigma|_\gamma$ near x.*

Similarly checked as previous statements.

So a connection gives a map of sheaves, (C^∞-vector fields) $\times$ $(C^\infty E) \to (C^\infty E)$. It's a map of C^∞-modules in the first coordinate.

2. Parallel transport

Given $E \to X$ smooth vector bundle, connection ∇, path $\gamma : [0,1] \to X$ with $\dot{\gamma} \neq 0$. Given a section section σ of γ^*E (this is a vector bundle on [0,1]). We can talk about $\nabla_{\dot{\gamma}}\sigma$ by the earlier discussion. We say that σ is ∇-constant if $\nabla_{\dot{\gamma}}\sigma = 0$.

Lemma. *Given X, E, ∇ and γ as before and $V \in E_{\gamma(0)}$, there exists a unique ∇-constant section of γ^*E such that $\sigma(\gamma(0)) = v$.*

The reason is that the required condition is a first order ODE.

With E, X, ∇, γ and v as above, $\sigma(\gamma(1))$ is called the parallel transport of v from $\gamma(0)$ to $\gamma(1)$, along γ with respect to ∇

Vocabulary: If γ is a contractible loop, parallel transport along γ can be a nontrivial endomorphism of $E_{\gamma(0)}$. It's called "holonomy around γ". If a connection has trivial holonomy for all contractible γ, then ∇ is called integrable or flat. If the connection is integrable, we get a map $\pi_1(X, x) \to GL(E_x)$. This map is called monodromy.

There is one more useful definition we often want: A connection is a map of sheaves

$$\nabla : C^\infty E \to \Omega^1 \otimes C^\infty E$$

such that:

$$\nabla(\sigma + \tau) = \nabla(\sigma) + \nabla(\tau)$$
$$\nabla(f\sigma) = df \otimes \sigma + f\nabla(\sigma)$$

The relationship between this definition and definition 2 is

$$\nabla_V(\sigma) = \langle V, \nabla(\sigma) \rangle$$

3. Connections in Coordinates

Let $X = U \subset R^n$ and $E \cong U \times R^m$ with the obvious vector bundle sturcutre. Then ∇ is determined by $\nabla_{\partial/\partial x_i}(e_j)$, everything else by linearity and Leibnitz and there's no restriction on what these should be.

Let $\nabla_{\partial/\partial x_i} e_a = \sum c^b_{ia} e_b$, where c^b_{ia} are nr^2 smooth functions. In other words, $\nabla e_a = \sum \sum c^b_{ia} e_b dx_i$. Then the Liebnitz and linearity conditions force

$$\nabla \left(\textstyle\sum_a f_a e_a \right) = \textstyle\sum_i df_a \otimes e_a + \sum_a f_a \left(\sum_i \sum_b c^b_{ia} dx_i \otimes e_b \right).$$

The above equality can also be written as:

$$\nabla\sigma = \text{``}d\sigma\text{''} + \left(\sum c^b_{ia} dx_i \right) \otimes \sigma$$

where $d\sigma$ is abusive notation which only makes sense in a given trivialization. Here $\left(\sum c^b_{ia} dx_i\right)$ is a $r \times r$ matrix whose entries are 1-forms. Given ∇, ∇', connections on E, $\nabla(\sigma) - \nabla'(\sigma)$ is of the form (matrix)$\otimes \Omega^1$. Dropping out of coordinates, $\nabla - \nabla'$ is a well-defined element in End$(E) \otimes \Omega^1$.

Suppose we have E, X and $\nabla : C^\infty E \to C^\infty E \otimes \Omega^1$. By linearity, we get a map $\nabla : C^\infty E \otimes \Omega^k E \to C^\infty E \otimes \Omega^{k+1}$.

$$\nabla : \sum \sigma_i \otimes d\omega_i = \sum \sigma_i \otimes \omega_i + \sum [\nabla\sigma_i \otimes \omega_i]$$

On the right hand side, the first term is already in $C^\infty E \otimes \Omega^{k+1}$, the bracket on the second term means the contraction map $\Omega^1 \otimes \Omega^k \to \Omega^{k+1}$.

We have to check that this is well defined. Specifically, we have $\sigma \otimes (f \cdot \omega) = (f\sigma) \otimes \omega$, so we have to check that we get the same result when we apply ∇ to either way of associating the tensor product.

$$\nabla\left(\sigma \otimes (f\omega)\right) = \sigma \otimes (df \wedge \omega) + \sigma \otimes (fd\omega) + f[\nabla\sigma \otimes \omega] =$$
$$\sigma \otimes (df \wedge \omega) + f\sigma \otimes (d\omega) + f[\nabla\sigma \otimes \omega]$$

$$\nabla\left((f\sigma) \otimes \omega\right) = f\sigma \otimes d\omega + [\nabla(f\sigma) \otimes \omega] = f\sigma \otimes d\omega + [\sigma \otimes df \otimes \omega] + [f\nabla\sigma \otimes \omega] =$$
$$f\sigma \otimes d\omega + \sigma \otimes (df \wedge \omega) + f[\nabla\sigma \otimes \omega]$$

The above computation tells us that ∇ is well-defined on the tensor product over C^∞-functions. So we have maps:

$$C^\infty E \to C^\infty E \otimes \Omega^1 \to C^\infty E \otimes \Omega^2 \to ...$$

Lemma: ∇^2 is 0 for all $C^\infty E \otimes \Omega^k \to C^\infty E \otimes \Omega^{k+1} \to C^\infty E \otimes \Omega^{k+2}$ if and only if $\nabla^2 : C^\infty E \to C^\infty E \otimes \Omega^2$ is 0.

Definition. *If this happpens, ∇ is called integrable.*

Another using perspective: ∇ is integrable if and only if for any commuting vector fields V, W, $\nabla_V \nabla_W = \nabla_W \nabla_V$ as maps $C^\infty E \to C^\infty E$. And, as mentioned before, ∇ is integrable if and only if the holonomy around every contractible loop is trivial.

4. From local systems to connections, from holomorphic vector bundles to $\overline{\partial}$-connections

Given $E \to X$ a local system, I'll define a connection on $C^\infty E$.

Take trivializations U_i on which E is just $U_i \times R^r$ and for which the transition maps are locally constant. I define $\nabla\sigma = d\sigma$.

We need to check well-definedness. Say on the intersection of open sets U_{ij} are related by $g_{j\leftarrow i}$ locally constant. Given σ in i-chart, member of j-chart is $g_{j\leftarrow i}\sigma$.

We have

$$d(g_{j\leftarrow i}\sigma) = (dg_{j\leftarrow i})\sigma + g_{j\leftarrow i}d\sigma = g_{j\leftarrow i}d\sigma$$

where the first term vanishes because $g_{j\leftarrow i}$ is locally constant and, thus, $dg_{j\leftarrow i} = 0$.

So this formula gives a well-defined section of $C^\infty E \otimes \Omega^1$. Moreover $\nabla^2 = 0$ because $d^2 = 0$.

We can now do something analogous with holomorphic vector bundles.

Given X complex manifold, $E \to X$ smooth complex vector bundle, we define a $\bar{\partial}$-connection to be a map $\nabla : C^\infty E \to C^\infty E \otimes \Omega^{0,1}$ such that:

$$\nabla(\sigma + \tau) = \nabla(\sigma) + \nabla(\tau)$$
$$\nabla(f\sigma) = \sigma \otimes \bar{\partial} f + f\nabla\sigma$$

Given $V = \sum f_i \partial/\partial\bar{z}_i$ we define $\nabla_V\sigma = \langle V, \nabla(\sigma)\rangle$. So a $\overline{\partial}$-connection only lets us take derivatives by $(0,1)$-vector fields.

Given a complex manifold X and $E \to X$ holomorphic vector bundle, define a $\bar{\partial}$-connection by:

(1) Passing to charts where $E \cong U_i \otimes C^r$
(2) Defining $\nabla\sigma = \bar{\partial}\sigma$

This is well-defined since $\bar{\partial} g = 0$ by the holomorphic condition. And as before, $\nabla^2 = 0$. This is exact, because exactness is a local concept and we have already proven the Dolbeault lemma.

In particular, doing this with $E = \mathcal{H}^p$ we get

$$0 \to \Omega^{p,0} \to \Omega^{p,1} \to \Omega^{p,2} \to ...$$

This is the Dolbeault complex. We now know how to do this for any holomorphic vector bundle.

Let's summarize:

For local systems:
- The kernel of $\nabla : C^\infty E \to C^\infty E \otimes \Omega^1$ is locally constant functions.
- The deRham complex $0 \to LC \to C^\infty E \to C^\infty E \otimes \Omega^1 \to$ is exact.
- For $q > 0$, we have $H^q(C^\infty E \otimes \Omega^p) = 0$, because of partitions of unity.
- So $H^q_{DR}(E) = H^q(LC(E))$.

For holomorphic vector bundles:
- The kernel of $\nabla : C^\infty E \to C^\infty E \otimes \Omega^{0,1}$ is holomorphic functions.
- The Dolbeault complex $0 \to \mathcal{H}ol(E) \to C^\infty E \to C^\infty E \otimes \Omega^{0,1} \to$ is exact.
- For $q > 0$, we have $H^q(C^\infty E \otimes \Omega^{0,p}) = 0$, because of partitions of unity.
- So $H^q_{Dolb}(E) = H^q(\mathcal{H}ol(E))$.