NOTES FOR 24 FEBRUARY 2011

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1. Terminological Clarifiation

Let's start by clearing up and making precise some abuses of terminology.

Suppose $E, F \to X$ are two smooth vector bundles over X. Consider the sheaves $C^\infty E, C^\infty F$, the sheaves of smooth sections, which are C^∞-modules. We can tensor them as modules to get $C^\infty E \otimes_{C^\infty} C^\infty F$, another sheaf of C^∞-modules. A section of this tensor product over some open set U is a formal sum

$$(C^\infty E \otimes_{C^\infty} C^\infty F)(U) = \left\{ \sum f_i(\sigma_i \otimes \tau_i) \right\} \Bigg/ \begin{array}{l} (\sigma_1 + \sigma_2) \otimes \tau = \sigma_1 \otimes \tau + \sigma_2 \otimes \tau \\ \sigma \otimes (\tau_1 + \tau_2) = \sigma \otimes \tau_1 + \sigma \otimes \tau_2 \\ f(\sigma \otimes \tau) = (f\sigma) \otimes \tau = \sigma \otimes (f\tau) \end{array} .$$

We can also tensor the vector bundles: $E \otimes F$ is a smooth vector bundle, and we have a natural isomorphism

$$C^\infty(E \otimes F) \cong C^\infty E \otimes_{C^\infty} C^\infty F.$$

To describe the map from right to left, given an element $\sigma \otimes \tau$ of $C^\infty E \otimes_{C^\infty} C^\infty F$, send it to the section $\sigma \otimes \tau$ of $C^\infty(E \otimes F)$ (check that this obeys the relations!). On charts where we have a trivialization, it is easy to see that we have an isomorphism, and isomorphism of sheaves is a local property.

We will switch back and forth between these two ways of thinking. For example, let $\nabla : E \otimes \Omega^1 \to E \otimes \Omega^2$ be a connection, sending $\sigma \otimes \omega \mapsto [\nabla\sigma \otimes \omega] + \sigma \otimes d\omega$. Locally, a connection on line bundles is

$$\sigma \mapsto d\sigma + \sigma\omega$$

for some $\omega \in \Omega^1$. Should we or should we not write $\sigma \otimes \omega$ instead? It depends on which way we're thinking!

When is ∇^2 zero? In the line bundle case, identify a section σ with a function f. We have

$$\sigma \overset{\nabla}{\mapsto} d\sigma + \sigma\omega = \sum \frac{\partial f(x)}{\partial x_i} dx_i + f \cdot \omega.$$

Apply ∇ again yields

$$\begin{aligned} \nabla^2(\sigma) &= \sum \left[\left(\nabla \frac{\partial f}{\partial x_i} \right) \otimes dx_i \right] + \sum 0 + \left[\sum \frac{\partial f}{\partial x_i} dx_i \otimes \omega \right] + [f\omega \otimes \omega] + f dw \\ &= \left[\sum \frac{\partial^2 f}{\partial x_i \partial x_j} \otimes dx_j \otimes d_i \right] + \left[\sum \frac{\partial f}{\partial x_i} \omega \otimes dx_i \right] + \left[\sum \frac{\partial f}{\partial x_i} dx_i \otimes \omega \right] + [f\omega \otimes \omega] + f d\omega \\ &= 0 + 0 + 0 + f d\omega = f d\omega \end{aligned}$$

where the second zero results from two term canceling. We conclude that $\nabla^2 : f \mapsto f d\omega$, so it is linear in f, and zero (i.e., ∇ is integrable) if and only if $d\omega = 0$ (i.e., ω is closed).

If we run through this with an $r \times r$ matrix $\omega = (\omega_{ij})_{i,j}$ instead[1], most of the terms will cancel similarly, but we will be left with

$$f \mapsto f d\omega + [(\omega_{ij})(\omega_{ij})] \, f$$

where we tensor together elements of our matrix whenever we would multiply them. Because matrix multiplication is not commutative, the entries of the matrix may not be symmetric and hence may not cancel when bracketed.

Xin Zhou asks for an example with curvature. Here is the sphere S^2, with θ the longitude and ϕ the lattitude. In (θ, ϕ) coordinates, our connection will look something like $\begin{pmatrix} u \\ v \end{pmatrix} \mapsto$

[1] Suggestion: do this in coordinates the first time you try it!

$\begin{pmatrix} du \\ dv \end{pmatrix} + \begin{pmatrix} 0 & -\sin(\phi)d\theta \\ \sin(\phi)d\theta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$. Parallel transport around γ should be rotation by $2\pi -$ (area inside γ).

2. Bilinear Forms on Vector Bundles

Let $E \to X$ be a vector bundle. We are interested in maps

$$\langle \cdot, \cdot \rangle : E \times_X E \to \mathbb{R} \text{ or } \mathbb{C}.$$

We want these to be smooth in each coordinate: if σ, τ are smooth sections, then $\langle \sigma, \tau \rangle$ is a smooth function. We will always want

$$\langle \sigma_1 + \sigma_2, \tau \rangle = \langle \sigma_1, \tau \rangle + \langle \sigma_2, \tau \rangle$$
$$\langle \sigma, \tau_1 + \tau_2 \rangle = \langle \sigma, \tau_1 \rangle + \langle \sigma, \tau_2 \rangle.$$

When over $\mathbb{R}$, we of course want (for σ, τ sections and λ a function; or σ, τ vectors and λ a scalar)

$$\langle \lambda\sigma, \tau \rangle = \lambda \langle \sigma, \tau \rangle = \langle \sigma, \lambda\tau \rangle.$$

When over $\mathbb{C}$ we will always want sesquilinearity:

$$\langle \lambda\sigma, \tau \rangle = \lambda \langle \sigma, \tau \rangle$$
$$\langle \sigma, \lambda\tau \rangle = \overline{\lambda} \langle \sigma, \tau \rangle.$$

We may additionally want for $\mathbb{R}$-vector bundles:

- symmetric: $\langle \sigma, \tau \rangle = \langle \tau, \sigma \rangle$
- antisymmetric/skew-symmetric: $\langle \sigma, \tau \rangle = -\langle \tau, \sigma \rangle$
- positive definite: symmetric and, for all vectors u, $\langle u, u \rangle \geq 0$ with $\langle u, u \rangle = 0$ if and only if $u = 0$.

Correspondingly for $\mathbb{C}$-vector bundles, we may want:

- Hermitian: $\langle \sigma, \tau \rangle = \overline{\langle \tau, \sigma \rangle}$
- anti-Hermitian: $\langle \sigma, \tau \rangle = -\overline{\langle \tau, \sigma \rangle}$
- positive definite Hermitian: Hermitian and, for all vectors u, $\langle u, u \rangle \geq 0$ with $\langle u, u \rangle = 0$ if and only if $u = 0$.

In the smooth case (rather than the holomorphic or locally constant), we have the following:

Lemma. *Let X be a smooth manifold, $E \to X$ a smooth real or complex vector bundle. Then there exists a smooth positive definite symmetric/Hermitian inner product on E.*

Proof. Take a locally finite cover U_i on which E is trivial, a partition of unity ρ_i with $\text{supp}(\rho_i) \subset U_i$, and positive definite inner products $\langle \cdot, \cdot \rangle$ on $E\big|_{U_i}$. Then $\sum \rho_i \langle \cdot, \cdot \rangle$ is positive definite on E. □

Remark: It is not important that the ρ_i add up to 1; what is important is that their sum be positive everywhere, and that they be smooth.

Warning: holomorphic vector bundles do not necessarily have holomorphic inner products! They do have smooth vector bundles (by the above Lemma). There are two interesting phenomena with holomorphic E which can only happen because of this fact. First, $E^* \ncong E$, where the dual is defined by gluing together duals on trivializations. In other words, E^* has gluing data $\left(g_{j \leftarrow i}^{-1}\right)^T$. In the smooth (real) case, we do have $E^* \cong E$ via the isomorphism $u \mapsto \langle u, \cdot \rangle$. Also, in the holomorphic world, we can have vector bundles

$$0 \to E \xrightarrow{\alpha} F \xrightarrow{\beta} G \to 0$$

exact on every fiber, but $F \ncong E \oplus G$. In the smooth world, take a positive definite inner product on F, and define $G' \subset F$ to be the orthogonal complement of $\alpha(E)$ (check that G' is a vector bundle!). Then $F \cong E \oplus G'$ and $\beta\big|_{G'} : G' \xrightarrow{\cong} G$.

Question: Do cokernels exist for smooth vector bundles? For maps injective on every fiber, yes, but in general the "correct" cokernel may be a sheaf that isn't a vector bundle: consider $X = \mathbb{R}$, $E = X \times \mathbb{R}$, $u \mapsto x \cdot u$; the cokernel we want is a skyscraper sheaf.

3. A Basic Fact about Sesquilinear Forms

Given $E \to X$ a complex vector bundle and $\langle \cdot, \cdot \rangle : E \times_X E \to \mathbb{C}$ sesquilinear, write $\langle \cdot, \cdot \rangle = g - i\omega$, where $g : E \times_X E \to \mathbb{R}$ and $\omega : E \times_X E \to \mathbb{R}$ are bilinear forms. Then $\langle \cdot, \cdot \rangle$ can be recovered from g or from ω. Not all g and ω come from sesquilinear inner products.

Given $\langle \cdot, \cdot \rangle$, we see that

$$g(iu, iv) = \mathrm{Re}(\langle iu, iv \rangle) = \mathrm{Re}(\langle u, v \rangle) = g(u, v)$$

and similarly, $\omega(iu, iv) = \omega(u, v)$. Hence sesquilinear forms yield complex-multiplication-invariant bilinear forms g and ω. If $\langle \cdot, \cdot \rangle$ is Hermitian, then we have

$$g(v, u) + i\omega(v, u) = g(u, v) - i\omega(u, v)$$

so g is symmetric and ω antisymmetric.

In the other direction, given a complex vector bundle E and a bilinear form $g : E \times_X E \to \mathbb{R}$ such that $g(iu, iv) = g(u, v)$. We claim that g is the real part of a unique $\langle \cdot, \cdot \rangle$. For uniqueness, suppose $g = \mathrm{Re}(\langle \cdot, \cdot \rangle)$. Then $\langle iu, v \rangle = i\langle u, v \rangle$, so

$$g(iu, v) - i\omega(iu, v) = i(g(u, v) - i\omega(u, v))$$

and hence $\omega(u, v) = g(iu, v)$. We conclude that g determines ω and hence $\langle \cdot, \cdot \rangle$ uniquely.

For existence, given such a g, define

$$\langle u, v \rangle = g(u, v) - ig(iu, v).$$

This formula has the correct linearity properties to be an inner product (check!).

Similarly, given ω such that $\omega(iu, iv) = \omega(u, v)$, there is a unique $\langle \cdot, \cdot \rangle$ of the form $g - i\omega$ by setting $g(u, v) = \omega(u, iv)$.

Moreover, $\langle \cdot, \cdot \rangle$ is Hermitian if and only if g is symmetric if and only if ω is antisymmetric. $\langle \cdot, \cdot \rangle$ is positive definite if and only if g is positive definite. By definition, we also say ω positive definite or positive (despite being antisymmetric). Explicitly, this means $\omega(u, iu) > 0$ for $u \neq 0$.

4. A Useful Lemma

Lemma. *Let E have a positive definite inner product $\langle \cdot, \cdot \rangle$ (real or complex). Let U be a neighborhood where E can be trivialized. Then E can be trivialized on U such that $\langle \cdot, \cdot \rangle$ is the standard inner product. (Explicitly, in the real case, $\langle (x_1, ..., x_r), (y_1, ..., y_r) \rangle = \sum x_i y_i$, and in the complex case, $\langle (x_1, ..., x_r), (y_1, ..., y_r) \rangle = \sum x_i \overline{y_i}$.)*

Proof. Gram-Schmidt preserves smoothness. □

5. Connections Again

Let ∇ be a connection on E. We say that ∇ preserves some inner product $\langle \cdot, \cdot \rangle$ if

- (motivation) for any curve γ and ∇-constant sections σ, τ on γ, we have $d\langle \sigma, \tau \rangle = 0$. In other words, parallel transport preserves this inner product.
- (definition) for all vector fields X, we have

$$X\langle \sigma, \tau \rangle = \langle \nabla_X \sigma, \tau \rangle + \langle \sigma, \nabla_X \tau \rangle.$$

  Abusing notation, we can equivalently write $d\langle \sigma, \tau \rangle = \langle \nabla \sigma, \tau \rangle + \langle \sigma, \nabla \tau \rangle$.

Let's write this in coordinates. Let $\langle \cdot, \cdot \rangle$ be a positive definite inner product. On U with basis e_i for E such that $\langle \cdot, \cdot \rangle$ is standard, write (with successively fewer and fewer coordinates)

$$\nabla_{\frac{\partial}{\partial x_i}} e_a = \sum c_{ia}^b e_b$$
$$\nabla_{\frac{\partial}{\partial x_i}} \sigma = \partial \sigma / \partial x_i + C_i \cdot \sigma$$
$$\nabla \sigma = d\sigma + \sum C_i dx_i \cdot \sigma$$

(where C_i is an $r \times r$ matrix). Then ∇ preserves $\langle \cdot, \cdot \rangle$ if

- Real case: $c_{ia}^b = -c_{ib}^a$; i.e., $C_i^T = -C_i$.
- Complex case: $c_{ia}^b = -\overline{c_{ib}^a}$; i.e., $\overline{C_i^T} = -C_i$.

A good and important exercise: what happens if we write this in the dz_i and $d\overline{z_i}$ coordinates?