NOTES FOR FEBRUARY 3

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We are headed towards proving:

If U a polydisc, $W \subset U$ a closed smooth $\mathbb{C}$-submanifold of dimension d then the topological cohomology $H^k_{top}(W, \mathbb{C})$, with coeffficients in $\mathbb{C}$, is the cohomology of the complex :

$$0 \to \mathcal{H}^0(W) \to \mathcal{H}^1(W) \to \mathcal{H}^2(W) \to \cdots \to \mathcal{H}^d \to 0,$$

where the above sequence is the De Rham complex using holomorphic forms.

We did this computation with $\mathbb{C} - \{0\}$ earlier.

Exactness of the Holomorphic De Rham Complex on a Smooth Manifold

Here is a fact which we were close to in a previous lecture, but never got around to proving:

Proposition 1. *For any smooth $\mathbb{C}$-submanifold of dimension d, the sequence*

$$0 \to \mathcal{LC}_{\mathbb{C}} \to \mathcal{H}^0 \to \mathcal{H}^1 \to \cdots \to \mathcal{H}^d \to 0$$

is an exact sequence of sheaves.

Proof. Exactness for sheaves is a local thing, so it suffices to check this on a polydisc $U_1 \times U_2 \times \cdots \times U_n$. Suppose we are given an exact p-form $\omega = \sum_I f_I dz_{i_1} \wedge dz_{i_2} \wedge \cdots \wedge dz_{i_p}$, where the f_I's are holomorphic on the polydisc and $\partial\omega = 0$. We argue similarly as we have done previously: Prove the claim by induction on the largest index, k, for which dz_k has a nonzero coefficient in $\partial\omega$. The base case is when $k = 0$, in which case there is clearly a preimage, namely 0.

For $\ell > k$, consider the coefficient of $dz_l \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ in $\partial\omega$, which is by definition $\frac{\partial f_I}{\partial z_\ell}$, so that by assumption $\frac{\partial f_I}{\partial z_l} = 0$ for all I. Hence, f_I is independent of $z_{k+1}, z_{k+2}, \ldots, z_d$.

Now let

$$\alpha := \sum_I \pm \left(\int_0^{z_k} f_I(z_1, \ldots, z_k) \right) dz_{i_1} \wedge \ldots dz_{i_{r-1}} \wedge \widehat{dz_k} \wedge dz_{i_{r+1}} \cdots \wedge dz_{i_p}.$$

There is no path dependence on the integral inside this definition since we are working on a polydisc. The $\pm$ signs are tricky to figure out but are unimportant. The fundamental theorem of calculus guarantees that $\partial\alpha$ and ω agree on all of their terms in which dz_k shows up, so that $\omega - \partial\alpha$ has no dz_ℓ terms, $\ell \geq k$. By induction, we have

$$\omega - \partial\alpha = \partial\beta$$

for some holomorphic $(p-1, 0)$-form. This shows $\omega = \partial(\alpha + \beta)$, so that ω is closed as desired. □

Now apply the preceding proposition to the case of a polydisc U with a closed submanifold W. The closed embedding $\iota \colon W \hookrightarrow U$ yields a pushforward

$$0 \to (\iota_* \mathcal{LC}_{\mathbb{C}})_W \to \iota_* \mathcal{H}^0_W(U) \to \iota_* \mathcal{H}^1_W(U) \to \cdots \to \iota_* \mathcal{H}^d_W(U) \to 0. \tag{1}$$

We know that $H^q(U, \iota_* \mathcal{LC}_{\mathbb{C}}) \cong H^q(W, \mathcal{LC}_{\mathbb{C}}) = H^q_{top}(W, \mathbb{C})$ where the first equality is a general fact about pushforwards of closed embeddings, and the last equality was mentioned earlier.

We also know that $H^0(U, \iota_* \mathcal{H}^p_W) \cong H^0(W, \mathcal{H}^p_W)$. We want to show that $H^q(U, (\iota_* \mathcal{LC}_{\mathbb{C}})_W)$ is computed by the cohomology of the complex (1). To see this, is suffices to show the higher cohomology of $\iota_* \mathcal{H}^p_W$ vanishes.

This leads us to a lemma which follows from problem 6 on Homework 3. Namely:

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Lemma 1. *If $\mathcal{E}$ is a sheaf of $\mathcal{O}$-mods and*

$$0 \to \mathcal{F}_N \to \mathcal{F}_{N-1} \to \cdots \to \mathcal{F}_0 \to \mathcal{E} \to 0$$

is an exact complex, where the $\mathcal{F}_i$ are free $\mathcal{O}$-mods and $H^q(U, \mathcal{O}) = 0$ for $q > 0$, then $H^q(U, \mathcal{E}) = 0$ for $q > 0$.

Proof. We will merely sketch the proof as it was done on the homework. First, note that the higher cohomlogy of a free $\mathcal{O}$-module will also vanish, as $H^q(U, \mathcal{O}^b) \cong H^q(U, \mathcal{O})^{\oplus b}$. To see the higher coholomology of $\mathcal{E}$ vanishes, let $Z_i := \mathcal{I}m(\mathcal{F}_i \to \mathcal{F}_{i-1})$ and observe that we have the following short exact sequences (one for each i):

$$0 \to Z_{i+1} \to \mathcal{F}_i \to Z_i \to 0.$$

Since the $\mathcal{F}_i$'s have vanishing higher cohomology, for $q > 0$, the LES in cohomology gives isomorphisms $H^q(U, Z_i) \cong H^{q+1}(U, Z_{i+1})$, and chaining these together gives isomorphsims

$$H^q(U, \mathcal{E}) \to H^{q+1}(U, Z_1) \to \cdots \to H^{q+N+1}(U, Z_{N+1}) = H^{q+N+1}(U, 0) = 0,$$

as desired.

When $q = 0$, similar reasoning gives an exact sequence on global sections:

$$0 \to H^q(U, \mathcal{F}_N) \to \cdots \to H^q(U, \mathcal{F}_0) \to H^q(U, \mathcal{E}) \to 0,$$

which we will be referencing later. □

In a few seconds, we will present a general situation– Cartan's Theorem –that will supply us with left resolutions by free $\mathcal{O}$-modules as in the hypotheses of the preceding lemma. However, we already can get our hands on some examples.

Example 1: Let W be a smooth hypersurface, $W = \{z \colon F(z) = 0\}$. One has an exact sequence of sheaves

$$0 \to \mathcal{O}_U \xrightarrow{\cdot F} \mathcal{O}_U \to \iota_* \mathcal{O}_W \to 0$$

where the left map is multiplication by F and the right map is restriction.

Note: To ensure that W is actually a (nonsingular) hypersurface, it suffices that one of the $\frac{\partial F}{\partial z_i}$'s is nonzero on W, in which case we have an implicit function theorem: Locally about any point for which $\frac{\partial F}{\partial z_1} \neq 0$, we can take as $(F, z_2, \ldots, z_n)$ coordinates

Example 2: If the dimension of U is 2 and the dimension of W is 1, we have

$$0 \to \mathcal{O} \xrightarrow{(\frac{\partial F}{\partial x} dx, \frac{\partial F}{\partial y} dy)} \mathcal{H}^1_U \to \iota_* \mathcal{H}^1_W \to 0.$$

The left map sends a holomorphic f to $f \frac{\partial F}{\partial x} dx + f \frac{\partial F}{\partial y} dy$. Note that the middle term is a free $\mathcal{O}$-module on two generators, dx and dy.

In particular, consider the case of a curve in a polydisc. For example, let

$$U = \{(z_1, z_2) \colon |z_1| < r_1,\ |z_2| < r_2\}$$

and W be defined by $F(z_1, z_2) = z_1 z_2 - 1$. So W is the annulus, $A = \{z \colon \frac{1}{r_2} < |z| < r_1\}$, after projecting onto z_1. Hence, $H^1(A, \mathcal{O}) = 0$.

Example 2': The preceding example has a generalization to higher dimensions. Let W be a smooth codimension r mandifold defined by $F_1 = \ldots F_r = 0$. One has a similarly explicit resolution:

$$0 \to \mathcal{O} \to \cdots \to \mathcal{O}^{\binom{r}{3}} \to \mathcal{O}^{\binom{r}{2}} \to \mathcal{O}^r \to \mathcal{O} \to \mathcal{O}_W \to 0.$$

Here, the maps $\mathcal{O}^{\binom{r}{k}} \to \mathcal{O}^{\binom{r}{k-1}}$ are as follows. We will think of them as left multiplication by a matrix whose columns are indexed by k-subsets of $[r] = \{1, 2, \ldots r\}$ and whose rows are indexed by $(k-1)$-subsets of $[r]$. The column corresponding to a k-subset, $I = \{i_1, i_2 \ldots i_k\}$ with the i_j's increasing, will have a nonzero entry in the rows corresponding to the $(k-1)$-subsets

$$I - \{i_j\},\ j = 1, 2 \ldots k.$$

Furthermore, the entry in the row corresponding to j will be $(-1)^j F_{i_j}$.

Coherent Sheaves and Cartan's Thoerems

Now we state Cartan's Theorem. First, a definition:

Definition 1. *For a complex manifold X and sheaf $\mathcal{E}$ of $\mathcal{O}$ modules, we call X and $\mathcal{E}$* **coherent** *if there exists an open cover $\{U_i\}$, and on each U_i, an exact sequence of sheaves:*

$$\mathcal{O}_{U_i}^{b_i(U_i)} \to \mathcal{O}_{U_i}^{b_0(U_i)} \to \mathcal{E}|_{U_i} \to 0$$

(i.e., the sheaf is "finitely presented" locally).

Theorem 1. *(Cartan) Suppose we are given U an open polydisc, K a closed subpolydisc, and $\mathcal{E}$ a coherent sheaf on U. Then we can shrink U to some open V, with $K \subset V \subset U$, such that $\mathcal{E}$ has a resolution*

$$0 \to \mathcal{O}^{b_n} \to \mathcal{O}^{b_{n-1}} \to \dots \mathcal{O}^{b_0} \to \mathcal{E} \to 0$$

on V. Here, n is the dimension of U.

To get around the technical assumption on V, note that applying the above repeatedly we obtain a sequence of open polydiscs $V_0 \subset V_1 \subset \dots$ with $U = \bigcup_i V_i$. Furthermore, $H^q(V_i, \mathcal{E}) = 0$ for $q > 0$, so that we may perform the Cech computation with respect to this open cover. We will skip this computation for now.

By the Lemma, on each V_i, the surjection of sheaves $\mathcal{O}_{V_i}^{b_0} \to \mathcal{E} \to 0$ gives a surjection of global sections $\mathcal{O}(V_i)^{b_0} \to \mathcal{E}(V_i) \to 0$.

Let the second map in the above be right multiplication by $(f_1, \dots, f_{b_0})$. Let $x \in V_i$. Since the map is surjective, these f_i's induce a surjective map on stalks $\mathcal{O}_x^{b_0} \to \mathcal{E}_x$. In other words, on each V_i, there exists global sections $f_1, \dots, f_{b_0}$ of $\mathcal{E}$ such that the $\mathcal{O}_x$-module spanned by the f_i's is $\mathcal{E}_x$. This shows $\mathcal{E}|_{V_i}$ is "globally generated" at x. One can show $\mathcal{E}$ is globally generated everywhere. This is part of the following:

Theorem 2:
(Cartan Theorem A) *On a polydisc, every coherent sheaf is globally generated.*
(Cartan Theorem B) *On a polydisc, every coherent sheaf has vanishing H^q, $q > 0$.*

In order to apply the theorem, note that if W is any complex subvariety (not necessarily assumed to be smooth), $\mathcal{O}_W$ is coherent. Here $\mathcal{O}_W$ means functions that are holomorphic in some neighborhood of W. Furthermore, $\mathcal{K}er, \mathcal{C}o\mathcal{K}er$ and $\mathcal{I}m$ of maps between coherent sheaves are coherent. These two facts in conjunction generate lots of examples of coherent sheaves.

Looking Ahead

We previously defined polydiscs to be $U_1 \times U_2 \times \dots \times U_n$ where each U_i was an open disc. By the Riemann mapping theorem, an nonempty simply connected proper subset of $\mathbb{C}$ is biholomorphic to such a disc, so that the theory carries over equally well if we take a product of such spaces. (We also allow the subsets to be all of $\mathbb{C}^n$, since we have been allowing the radius of the polydisc to be infinite). In particular, we are allowed to take products of rectangles, as we will be doing in the upcoming lectures in which we prove Cartan's theorem for $\iota_* \mathcal{H}^p$.

Here is a crude sketch of how we will prove the theorem. Given such a rectangle, subdivide it into smaller rectangles until one of $\frac{\partial f}{\partial x_i} \neq 0$ on each rectangle. This implies W is of the form $F_1 = F_2 = \dots = F_r = 0$ for some F_i's. Then the Koszul resolution resolves $\iota_* \mathcal{H}^p W$ on each box. We will then show that given two such resolutions of length k on adjacent boxes, we can glue them together to get a resolution on the union.

In particular, we will need to do some analysis for this specific case of gluing: suppose we are given 2 adjacent boxes and $\mathcal{E}$ on $U \cup V$ such that on open neighborhoods of U and V we have $\mathcal{O}^b|_U \cong \mathcal{E}|_U$, $\mathcal{O}^b|_V \cong \mathcal{E}|_U$. Then $\mathcal{E} \cong \mathcal{O}^b$ on all of $U \cup V$. Once we establish this claim, the rest is all algebra.

One warning: The most obvious guess is that given $W \subset U$, a codimension r subvariety, then there exists a free resolution of length r. In fact, the length of the resolution is $\geq$ to the codimension, and it is equal for all subvarieties if and only if $\mathcal{O}_W$ (the structure sheaf of W) is Cohen-Macaulay.