NOTES FOR FEBRUARY 8

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Lemma (Cartan)**.** *Let K, L be closed polyboxes in $\mathbb{C}^n$ with a common side, U, V open neighborhoods of K, L respectively. Given $H : U \cap V \to \mathrm{GL}_r(\mathbb{C})$ holomorphic in each coordinate then (after possibly shrinking U and V to U' and V', still containing K and L) there are holomorphic maps $F : U' \to \mathrm{GL}_r(\mathbb{C})$, $G : V' \to \mathrm{GL}_r(\mathbb{C})$ such that $H = F^{-1}G$.*

Proof when $r = 1$. Let $h = \log H$ (this is ok since $U' \cap V'$ can be made simply connected after the shrinking). Look at the Čech complex

$$0 \to \mathcal{O}(U) \oplus \mathcal{O}(V) \to \mathcal{O}(U \cap V) \to 0.$$

This computes $H^1(U \cup V, \mathcal{O})$ which we already know to be 0. So there is some $(f, g) \in \mathcal{O}(U) \oplus \mathcal{O}(V)$ such that $g - f = h$. Let $G = e^g, F = e^f$, then $H = F^{-1}G$. We'll do something similar for the general case, but we need to be careful with taking log and exp of matrices. □

1. A quantitative Čech vanishing

We are now going to reprove the above result, keeping track of the size of everything. Specifically we'll show:

The next lemma is substantially corrected from the version presented in class. I couldn't quite make the proof in Hugo and Rossi work; so here is my best fix. David Speyer

Lemma. *Given K, L, U, V as above. There are chains of open polyboxes $U \supsetneq U' \supsetneq U'' \supsetneq K$ and $V \supsetneq V' \supsetneq V'' \supsetneq L$ and a constant $\in \mathbb{R}_{>0}$ such that given $h \in \mathcal{O}(U \cap V)$ with $|h| < M$ on $U' \cap V'$, there are $f \in \mathcal{O}(U''), g \in \mathcal{O}(V'')$ such that $h = g - f$ with $|g|, |f| < CM$.*

Proof. Let Z be the sheaf of $\bar{\partial}$-closed $(0, 1)$-forms so we have the exact sequence

$$0 \to \mathcal{O} \to C^\infty \to Z \to 0$$

which induces the exact sequence of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(U) \oplus \mathcal{O}(V) & \longrightarrow & C^\infty(U) \oplus C^\infty(V) & \longrightarrow & Z(U) \oplus Z(V) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}(U \cap V) & \longrightarrow & C^\infty(U \cap V) & \longrightarrow & Z(U \cap V) & \longrightarrow & 0 \end{array}.$$

We start with $h \in \mathcal{O}(U \cap V)$, and we want to lift it to $\mathcal{O}(U) \oplus \mathcal{O}(V)$. The first step is to map h over to $C^\infty(U \cap V)$, meaning to consider the same function as a smooth function. We then lift h to $C^\infty(U) \oplus C^\infty(V)$ "using partitions of unity". Let's be explicit about how we do that:

Choose $\sigma : \mathbb{R} \to [0, 1]$ such that $\sigma(x_1) = 0$ on $U \setminus V$ and $\sigma(x_1) = 1$ on $V \setminus U$ (and σ is smooth) where x_1 is the first coordinate in C^n and the common face of K and L is a hyperplane with constant x_1 coordinate. We can then define new functions $g_0 = (1 - \sigma(x_1))h$ and $f_0 = -\sigma(x_1)h$ which can be extended to V and U respectively by 0. These satisfy $|g_0|, |f_0| < M$. Now (f_0, g_0) lives in $C^\infty(U) \oplus C^\infty(V)$. Unfortunately, f_0 and g_0 are probably not holomorphic. The map $\overline{\partial}$ to $Z(U) \oplus Z(V)$ measures our failure to take f_0 and g_0 holomorphic.

On $U \cap V$, $g_0 - f_0 = h$ so $\partial g_0/\partial \bar{z}_1 - \partial f_0/\partial \bar{z}_1 = 0$ where z_1 is the first coordinate. Thus we can define a C^∞ function F by $F = \partial g_0/\partial \bar{z}_1$ on U and $F = \partial f_0/\partial \bar{z}_1$ on V. On $U \cap V$, we have $F = (\partial \sigma/\partial \bar{z}_1)h$ because h is holomorphic; outside of $U \cap V$, we have $F = 0$ since it is the derivative of the zero function. So $|F| \leq C_1 M$ for some constant C_1, namely, the maximum of the continuous function $|\partial \sigma/\partial \bar{z}_1|$.

Choose chains of open polyboxes $U \supsetneq U' \supsetneq U'' \supsetneq K$ and $V \supsetneq V' \supsetneq V'' \supsetneq L$ and a smooth hat function τ such that: $\tau = 1$ on $U'' \cup V''$, and τ is 0 on $(U \cup V) \setminus (U' \cup V')$, and $0 \leq \tau \leq 1$.

Define the function θ on $U \cup V$ by

$$\theta(z) = \frac{1}{2\pi i} \int_{U \cup V} \frac{F(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} \tau(\zeta) d\text{Area}$$

where $z = (z_1, \ldots, z_n)$.

When we proved Dolbeault's lemma (January 27), we showed that this integral converges and $\partial \theta/\overline{\partial} z_1 = F$ on $U'' \cup V''$. We want to bound the size of θ. We have the simple bound

$$|\theta(z_1)| \leq \frac{1}{2\pi} \int_{U' \cup V'} \frac{C_1 M}{|z_1 - \zeta|} d\text{Area}.$$

(Since τ is zero outside $U' \cup V'$, we can use the bound $|F| \leq C_1 M$, which was proved on $U' \cup V'$.)

Choose some radius R large enough that, for any $z_1 \in U' \cup V'$, the box $U' \cup V'$ is contained in the disc of radius R around z_1. Then our bound is

$$\leq \frac{1}{2\pi} \int_{B(R, z_1)} \frac{C_1 M}{|z_1 - \zeta|} d\text{Area}.$$

Switching to polar coordinates, this is

$$\frac{1}{2\pi} \int_{r=0}^{R} \int_{\theta=0}^{2\pi} \frac{C_1 M}{r} r dr d\theta = \int_{r=0}^{R} C_1 M dr = C_1 R M.$$

So, taking $C_2 = C_1 R$, we have the bound $|\theta| < C_2 M$ on $U' \cup V'$ and $\partial \theta/\overline{\partial} z_1 = F$ on $U'' \cup V''$.

Now, set $f = f_0 - \theta$ and $g = g_0 - \theta$. Since $\partial \theta/\partial \bar{z}_1 = F$ we will have $\partial f/\partial \bar{z}_1 = \partial g/\partial \bar{z}_1 = 0$. This makes f and g holomorphic because f_0, g_0 and θ are holomorphic in the other variables. In addition,

$$|f| \leq |f_0| + |\theta| \leq M + C_2 M = CM$$

for some constant C. The same argument works for g so we have f, g with $|f|, |g| < CM$ and $f - g = h$. $\square$

2. The matrix exponential

Recall the following: For A an $r \times r$ matrix, $|A| = \max_{|v|=1} |Av|$ (which exists by compactness). For $(v_1, \ldots, v_n) = v \in \mathbb{C}^r$, $|v| = (\sum |v_i|^2)^{1/2}$. Also

$$|AB| \leq |A||B|$$

$$|A+B| \leq |A| + |B|$$

$$|A_{ij}| \leq |A| \leq \sum_{i,j} |A_{ij}|.$$

For any matrix a we define $e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!}$ and for any A with $|A - \mathrm{Id}\,| < 1$, $\log A = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(A - \mathrm{Id})^k$. These satisfy $\exp(\log A) = A$. Also $|\exp(a) - \mathrm{Id}\,| = O(|a|)$ and $|\log(1 + A)| = O(|A|)$. Unfortunately, $e^{f+g} \neq e^f e^g$. The left hand side is

$$1 + f + g + (f+g)^2/2 + \cdots = 1 + f + g + f^2/2 + fg/2 + gf/2 + g^2/2 + \ldots$$

while the right hand side is

$$(1 + f + f^2/2 + \ldots)(1 + g + g^2/2 + \ldots) = 1 + f + g + f^2/2 + fg + g^2/2 + \ldots$$

and $fg \neq fg/2 + gf/2$.

The following lemma says there is a bound on how much e^{f+g} and $e^f e^g$ differ. Better bounds can be given that depend on the commutator of f and g, but this will be good enough for our purposes.

Lemma. *There is a constant $C > 0$ such that if $|f|, |g| < 1/10$ then*

$$|e^{f+g} - e^f e^g| \leq C|f||g|.$$

Proof.

$$\left| e^f e^g - \left(\mathrm{Id} + \sum_{k=1}^{\infty} \frac{f^k}{k!} + \sum_{k=1}^{\infty} \frac{g^k}{k!} \right) \right| = \left| \sum_{k,l \geq 1} \frac{f^k g^l}{k! l!} \right|$$

$$\leq \sum_{k,l \geq 1} \frac{|f|^k |g|^l}{k! l!} = (e^{|f|} - 1)(e^{|g|} - 1) \leq C_1 |f||g|$$

for some constant C_1. We have a similar inequality for e^{f+g}:

$$\left| e^{f+g} - \left(\mathrm{Id} + \sum_{k=1}^{\infty} \frac{f^k}{k!} + \sum_{k=1}^{\infty} \frac{g^k}{k!} \right) \right| = \left| \sum_{k,l \geq 1} \frac{1}{(k+l)!} \sum_{\substack{\text{orderings of} \\ k\ f\text{'s and } l\ g\text{'s}}} (fggfgff\ldots) \right|$$

$$\leq \sum_{k,l \geq 1} \frac{1}{(k+l)!} \sum_{\substack{\text{orderings of} \\ k\ f\text{'s and } l\ g\text{'s}}} |f|^k |g|^l = \sum_{k,l \geq 1} \frac{1}{(k+l)!} \frac{(k+l)!}{k! l!} |f|^k |g|^l$$

$$= (e^{|f|} - 1)(e^{|g|} - 1) \leq C_1 |f||g|.$$

Then we apply the triangle inequality and we win! □

We also need

Lemma (Runge)**.** *Given K a compact polybox in $\mathbb{C}^n$, U an open neighborhood about K, $f \in \mathcal{O}(U)$, and $\epsilon > 0$, there is a polynomial p such that $|f - p| < \epsilon$ on K.*

This theorem is obvious for disks because we can just cut off the Taylor series, but some trickiness is necessary to get it for any region. It will be proved in the homework.

3. Proving Cartan's Lemma

Remember that given $H : U \cap V \to \mathrm{GL}_r \mathbb{C}$ we want $H = F^{-1}G$ for some $F : U \to \mathrm{GL}_r \mathbb{C}$, $G : V \to \mathrm{GL}_r \mathbb{C}$.

Step 1: Given $\delta > 0$ we can shring U and V and write $H = H_0 H_1 \cdots H_N$ (where each $H_i : U \cap V \to \mathrm{GL}_r \mathbb{C}$) with $|H_i - \mathrm{Id}\,| < \delta$ for $i > 0$ and H_0 constant.

Shift coordinates so that $O \in U \cap V$. So for $z \in U \cap V$, $\alpha \in [0,1]$, αz is also in $U \cap V$. The idea is that

$$H(z) = H(0)(H(0)^{-1}H(\alpha_1 z))(H(\alpha_1 z)^{-1}H(\alpha_2 z)) \cdots (H(\alpha_{N-1} z)^{-1}H(z))$$

and this will give us what we want for suitable choices of α_i. We can choose $K \subset U' \subset K' \subset U$ and $L \subset V' \subset L' \subset V$ such that for any $\alpha \in [0,1]$ there is an interval (β_1, β_2) around α with $|H(\beta z)^{-1}H(\alpha z) - \mathrm{Id}\,| < \delta/100$ for $\beta \in (\beta_1, \beta_2)$ and $z \in K' \cap L'$. For α_1, α_2 in (β_1, β_2) we then have $|H(\alpha_1 z)^{-1}H(\beta_2 z) - \mathrm{Id}\,| < \delta$. By compactness of $[0,1]$ there are finitely many such intervals and we can take the α_i's to be in the overlaps of successive intervals.

Step 2: Let $h_i = \log H_i$. By Runge's lemma we can (after shrinking our neighborhoods again) find p_i, a collection of $r \times r$ matrices with polynomial entries, such that $|h_i - p_i|$ is very small. Note that e^{p_i} is defined and in $\mathrm{GL}_r \mathbb{C}$ everywhere. Replacing H by $H' = e^{-p_N} e^{-p_{N-1}} \cdots e^{-p_1} H(0)^{-1} H$, we have reduced to the case where $|H - \mathrm{Id}\,|$ is very small. This is because if $(F')^{-1}G = H'$ we can just multiply F' by $H(0)e^{p_1} \ldots e^{p_N}$ to get a solution for $F^{-1}G = H$.

Step 3: Let $h = \log H$ and let $h = f - g$ with $|f||g| < C_1 |h|$. Let $F_1 = e^f, G_1 = e^g, H_1 = F_1 H G_1^{-1}$. Note that

$$|H_1 - \mathrm{Id}\,| = |e^f e^{g-f} e^{-g} - \mathrm{Id}\,| = |e^f(e^{-f}e^g + O(|f||g|))e^{-g} - \mathrm{Id}\,|$$

$$= |O(|f||g|)| \leq C_2 |f||g| \leq C_2 C_1^2 |h|^2 \leq C_2 C_1^2 C_3^2 |H - \mathrm{Id}\,|^2.$$

(Here the C_3 is the constant in the bound $|h| = O(|e^h - 1|)$.)

Take H close enough to Id, so that $|H_1 - \mathrm{Id}\,| < \frac{1}{2}|H - \mathrm{Id}\,|$. Let $h_1 = \log H_1$, and let $h_1 = g_2 - f_2$, $F_2 = e^{f_2}$, $G_2 = e^{g_2}$, $H_2 = F_2 H_1 G_2^{-1}$ etc. Then $|F_k - \mathrm{Id}\,| = O(2^{-k})$ and $|G_k - \mathrm{Id}\,| = O(2^{-k})$ so $F := \cdots F_2 F_1$ and $G := \cdots G_2 G_1$ converge. Then

$$FHG^{-1} = \cdots F_2 F_1 H G_1^{-1} G_2^{-1} \cdots$$

$$= (\mathrm{Id} + O(2^{-k-1}))(\mathrm{Id} + O(2^{-k}))(\mathrm{Id} + O(2^{-k-1})) = \mathrm{Id} + O(2^{-k})$$

so $FHG^{-1} = \mathrm{Id}$.

4. A remark on matrix logarithms

Remark: After class, several students asked me a good question. Given U a simply connected domain in $\mathbb{C}^n$, and H a holomorphic function $U \to \mathrm{GL}_r \mathbb{C}$, is there a holomorphic logarithm h of H? Recall that this is true for $r = 1$: Even though the power series for log does not converge if H is far from 1, the simple connectedness of U still lets us define the logarithm.

The answer is **no**. I'll make life easier on myself and just show that, if we have $H(0) = \mathrm{Id}$, and require that $h(0) = 0$, there need not be a logarithm. I am reasonably sure you should be able to extend this example to work without pinning down h at a basepoint.

The key point is that the differential of the exponential map is not always invertible. Let J be the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Note that $J^2 = -\mathrm{Id}$ so we have

$$e^{\theta J} = \sum \frac{\theta^n J^n}{n!} = \sum_m (-1)^m \frac{\theta^{2m}}{(2m)!}\mathrm{Id} + \sum_m (-1)^m \frac{\theta^{2m+1}}{(2m+1)!} J = (\cos\theta)\mathrm{Id} + (\sin\theta)J.$$

In particular, notice that $e^{\pi J} = -\mathrm{Id}$.

We will show that the matrix exponential, as a map from 2×2 matrices to themselves, has noninvertible differential at πJ.

Proof that the matrix exponential has noninvertible differential at πJ. Let S be the set of matrices of the form $g(\pi J)g^{-1}$, for $g \in \mathrm{GL}_2(\mathbb{C})$. Explicitly, S is given by the equations $\mathrm{Tr} = 0$ and $\det = 1$, and is a two dimensional complex manifold. Now, for $g(\pi J)g^{-1}$ in S, we have

$$e^{g(\pi J)g^{-1}} = \sum \frac{\pi^n (gJg^{-1})^n}{n!} = g\left(\sum \frac{\pi^n J^n}{n!}\right)g^{-1} = g(-\mathrm{Id})g^{-1} = -\mathrm{Id}$$

where the second equality is by g's and (g^{-1})'s canceling.

So the matrix exponential collapses S to a point. So the differential of exp must vanish on the tangent plane to S at πJ. □

Now, choose M a 2×2 matrix such that the vector in direction M is not in the image of the differential of the exponential map at πJ. Look at the map $H : (x, y) \mapsto e^{xJ} + yM$. The image of this map is invertible in an open neighborhood U of $[0, \pi] \times \{0\}$ within $\mathbb{C}^2$. Now, suppose that H has a logarithm h on U with $h(0,0) = 0$. Then one can show that $h(\theta, 0) = \theta J$. Then, for small t, we are suppose to have $\exp(h(\theta, t)) = -\mathrm{Id} + tM$. But we chose M so that there is no smooth curve $f(t)$ with $f(0) = \pi J$ and $f(t) = -\mathrm{Id} + tM + O(t^2)$. So $h(\pi, t)$ is not smooth, and h is not holomorphic.