NOTES FOR JANUARY 13

TENGREN ZHANG

1. Definitions and Examples of Sheaves and Presheaves

Let X be a topological space. A ***presheaf*** $\mathcal{E}$ gives a set $\mathcal{E}(U)$ for every open set $U \subset X$ such that for every inclusion of open sets $V \subset U \subset X$, we have a map $\rho_V^U : \mathcal{E}(U) \to \mathcal{E}(V)$ called ***restriction*** that obeys the presheaf axioms, i.e. $\rho_U^U = \mathrm{Id}$ for any open set $U \subset X$ and $\rho_W^U = \rho_W^V \circ \rho_V^U$ for all open sets $W \subset U \subset V$.

A ***sheaf*** is a presheaf with the additional property that for every open set U and every open cover $\{U_i\}$ of U, if there are $f_i \in \mathcal{E}(U_i)$ such that $\rho_{U_i \cap U_j}^{U_i}(f_i) = \rho_{U_i \cap U_j}^{U_j}(f_j)$ for all pairs (i, j), then there exists a unique $f \in \mathcal{E}(U)$ such that $\rho_{U_i}^U(f) = f_i$ for all i. Note that if we start with an open cover $\{U_i\}$ of U and some $f \in \mathcal{E}(U)$, and define $f_i := \rho_{U_i}^U(f)$, then the presheaf condition ensures $\rho_{U_i \cap U_j}^{U_i}(f_i) = \rho_{U_i \cap U_j}^{U_i} \circ \rho_{U_i}^U(f) = \rho_{U_i \cap U_j}^U$, so for any pair (i, j), $\rho_{U_i \cap U_j}^{U_i}(f_i) = \rho_{U_i \cap U_j}^{U_j}(f_j)$.

Here are some examples of presheaves and sheaves:

1. Let X be a topological space, and for every open subset $U \subset X$, let $\mathcal{E}(U) = \mathbb{R}^U$, the set of real valued functions on U. Let ρ_V^U be restriction. This is a sheaf.
2. For every open subset $U \subset X$, if we let $\mathcal{E}(U)$ to be the continuous functions from U to $\mathbb{R}$ and ρ be restriction, then we also get a sheaf.
3. Let X be a smooth manifold and $U \subset X$ be open. Let $\mathcal{E}(U)$ be the set of smooth $\mathbb{R}$-valued functions on U and ρ be restriction. This again is a sheaf.
4. If X is a complex manifold or an open subset of $\mathbb{C}^n$, then the assignment of each open subset of X to the $\mathbb{C}$-valued $\mathbb{C}$-analytic functions on U, with ρ being restriction, is also a sheaf.
5. Let X be a smooth manifold, $\Omega^k(U)$ the smooth k-forms on U and ρ restriction. Ω^k is also a sheaf.
6. The ***closed k-forms*** are k-forms ω on U such that $d\omega = 0$. If X is a smooth manifold, the closed k-forms with the usual restriction form a sheaf.
7. The ***exact k-forms*** are k-forms ω on U such that there is some $(k-1)$-form η on U with $d\eta = \omega$. If X is a smooth manifold, the closed k-forms with the usual restriction form a presheaf but not a sheaf, because even if there is some η_i on each U_i in the open cover of U such that $d\eta_i = \omega|_{U_i}$, there need not been a global η on U such that $d\eta = \omega$.
8. The constant functions on any topological space X forms a presheaf, while the locally constant functions form a sheaf.
9. Let X be a topological space and choose $x \in X$. For any open set $U \subset X$, let
$$\mathcal{E}(U) = \begin{cases} \mathbb{R} & \text{if } x \in U \\ \{0\} & \text{if } x \notin U \end{cases}, \quad \rho_V^U = \begin{cases} \mathrm{Id} & \text{if } x \in V \subset U \\ 0 & \text{otherwise} \end{cases}.$$
This is a sheaf, known as the ***skyscraper sheaf***.

Let X be a topological space. A ***sheaf of abelian groups*** is a sheaf $\mathcal{E}$ that assigns to each open subset $U \subset X$ an abelian group, and each ρ_V^U is a morphism of groups. A ***sheaf of commutative rings*** is a sheaf that assigns to each open subset $U \subset X$ a commutative ring, and each ρ_V^U is a morphism of rings. In particular, a sheaf of abelian groups is a sheaf of commutative rings.

A ***ringed space*** is a topological space X with a sheaf $\mathcal{O}$ of commutative rings. With this ringed space, we can define a ***sheaf of $\mathcal{O}$-modules*** to be a sheaf $\mathcal{M}$ where $\mathcal{M}(U)$ is an $\mathcal{O}(U)$-module for all open $U \subset X$, and each $\rho_V^U : \mathcal{M}(U) \to \mathcal{M}(V)$ is an $\mathcal{O}(U)$-module morphism, i.e. for all $x, y \in \mathcal{M}(U)$, $\rho_V^U(x) + \rho_V^U(y) = \rho_V^U(x + y)$, and for all $f \in \mathcal{O}(U)$ and $x \in \mathcal{M}(V)$, $\rho_V^U(f \cdot x) = \rho_V^U(f) \cdot \rho_V^U(x)$. (Here, we notate both the module morphism $\mathcal{M}(U) \to \mathcal{M}(V)$ and the ring morphism $\mathcal{O}(U) \to \mathcal{O}(V)$ as ρ_V^U, but it should be clear from the context which map we are referring to.)

Examples of sheaves of rings include locally constant functions on topological spaces, smooth functions on smooth manifolds, analytic functions on analytic manifolds. Examples of sheaves of modules include k-forms (or sections of vector bundles) as modules of C^∞ functions, closed k-forms as modules of locally constant functions (on differentiable manifolds).

2. Maps of sheaves and sheafification

Let X be a topological space and $\mathcal{E}$, $\mathcal{F}$ two sheaves on X. A ***map of sheaves*** $\phi : \mathcal{E} \to \mathcal{F}$ is the data, for every open set $U \subset X$, of a map $\phi|_U : \mathcal{E}(U) \to \mathcal{F}(U)$ such that for every $V \subset U$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{E}(U) & \xrightarrow{\phi|_U} & \mathcal{F}(U) \\ \downarrow{\rho^U_V} & & \downarrow{\rho^U_V} \\ \mathcal{E}(V) & \xrightarrow{\phi|_V} & \mathcal{F}(V) \end{array}$$

Given a topological space X and a presheaf $\mathcal{E}$, for any point $x \in X$, the ***stalk*** $\mathcal{E}_x$ is the direct limit $\varinjlim_{U \ni x} \mathcal{E}(U) := (\coprod_{U \ni x} \mathcal{E}(U))/\sim$. Here, for $f \in \mathcal{E}(U)$, $g \in \mathcal{E}(V)$, $x \in U \cap V$, we define $f \sim g$ if there exists some open set $W \subset U \cap V$, $W \ni x$ such that $\rho^U_W(f) = \rho^V_W(g)$. Note that it is possible that W is smaller than $U \cap V$. You may have seen this concept in an analysis class in the notion of a ***germ***: A germ is an equivalence class of functions on a pointed topological space, where two functions are equivalent if they agree in some neighborhood of the marked point.

Let $\mathcal{E}$ and $\mathcal{F}$ be two sheaves on X and $\phi : \mathcal{E} \to \mathcal{F}$ be a map of sheaves. For any $f \in \mathcal{E}_x$, choose a representative $(U, \tilde{f})$ (i.e. $U \ni x$ is an open set and $\tilde{f} \in \mathcal{E}(U)$ represents f) and let $\tilde{g} = \phi_U(\tilde{f})$. One can check that if we chose another representative $(U', \tilde{f}')$ and let $\tilde{g}' = \phi_{U'}(\tilde{f}')$, then $\tilde{g} \sim \tilde{g}'$. Thus, ϕ induces a map of stalks $\phi : \mathcal{E}_x \to \mathcal{F}_x$.

Let ϕ be a map of sheaves. We say ϕ is ***injective*** if the induced map on stalks $\phi : \mathcal{E}_x \to \mathcal{F}_x$ is injective for all x. This is equivalent to $\phi_U : \mathcal{E}(U) \to \mathcal{F}(U)$ being injective for all open sets U. Similarly, ϕ is said to be ***surjective*** if $\phi : \mathcal{E}_x \to \mathcal{F}_x$ is surjective for every x. This is equivalent to saying that for every open set U and every $g \in \mathcal{F}(U)$, there is an open cover $\{U_i\}$ of U and $f_i \in \mathcal{E}(U_i)$ such that $\phi|_{U_i}(f_i) = g|_{U_i} = \rho^U_{U_i}(g)$.

Let $\mathcal{P}$ be a presheaf. The ***sheafification*** of $\mathcal{P}$ is the sheaf $\mathcal{S}$ where $\mathcal{S}(U)$ is the set of functions $f : U \to \coprod_{x \in U} \mathcal{P}_x$ such that $f(x) \in \mathcal{P}_x$ for all x and there is an open cover $\{U_i\}$ of U and $g_i \in \mathcal{P}(U_i)$ with the property that for all $y \in U_i$, g_i represents $f(y)$ in $\mathcal{P}_y$. One can check here that $\mathcal{P}$ is a sheaf with ρ being restriction, and that it satisfies the following universal property:

Theorem: Let $\mathcal{P}$ be a presheaf and $\mathcal{S}$ its sheafification. There is a map of presheaves $\mathcal{P} \to \mathcal{S}$ such that for any map of presheaves $\mathcal{P} \to \mathcal{E}$, where $\mathcal{E}$ is a sheaf, there exists a unique map of sheaves $\mathcal{S} \to \mathcal{E}$ that makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & \mathcal{E} \\ \downarrow & \swarrow & \\ \mathcal{S} & & \end{array}$$

Given a topological space X, let $\mathcal{E}$ and $\mathcal{F}$ be sheaves of abelian groups and let $\phi : \mathcal{E} \to \mathcal{F}$ be a map of sheaves. We define $\mathcal{K}er(\phi)$ be the presheaf that assigns each open set $U \subset X$ to the group $\mathrm{Ker}(\phi_U)$. It is a straightforward exercise to check that $\mathcal{K}er(\phi)$ is a sheaf with ρ as restriction. We also define $\mathcal{C}o\mathcal{K}er(\phi)$ to be the sheafification of the presheaf that assigns each open $U \subset X$ to $\mathrm{CoKer}(\phi_U)$ and $\mathcal{I}m(\phi)$ to be the sheafification of the presheaf that takes each U to $\mathrm{Im}(\phi_U)$. Note that ϕ is injective if and only if $\mathcal{K}er(\phi) = 0$ and ϕ is surjective if and only if $\mathcal{C}o\mathcal{K}er(\phi) = 0$ (i.e. they take every open U to the trivial group).

Sheaves of abelian groups form an ***abelian category***. See the appendix for more on this. Basically, it means that kernels, cokernels and images behave just like they do for abelian groups.

3. Exactness

A sequence of maps of sheaves of abelian groups $\mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\alpha} \mathcal{C}$ is said to be ***exact*** if $\mathcal{K}er(\beta) \simeq \mathcal{I}m(\alpha)$ or $\mathcal{C}o\mathcal{K}er(\alpha) \simeq \mathcal{I}m(\beta)$ as sheaves (These two conditions are equivalent). More explicitly, this sequence is exact if for every open set U and every $g \in \mathcal{B}(U)$ such that $\beta(g) = 0$, there is an open cover $\{U_i\}$ of U and elements $f_i \in \mathcal{A}(U_i)$ for all i satisfying $\alpha(f_i) = \rho^U_{U_i}(g)$.

Theorem: Suppose $0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\alpha} \mathcal{C} \to 0$ is exact. Then $0 \to \mathcal{A}(X) \to \mathcal{B}(X) \to \mathcal{C}(X)$ is exact.

Proof: First, we prove exactness at $\mathcal{A}(X)$. Suppose $f \in \mathcal{A}(X)$ such that $\alpha(f) = 0$, then $f \in \mathcal{K}er(\alpha)(X)$. Since $0 \to \mathcal{A} \to \mathcal{B}$ is exact, $\mathcal{K}er(\alpha)(X) = \{0\}$, which means $f = 0$. This proves exactness holds at $\mathcal{A}(X)$.

Next, we do the same for $\mathcal{B}(X)$. Suppose that $g \in \mathcal{B}(X)$ and $\beta(g) = 0$, then by the exactness of $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$, we know that there is an open cover $\{U_i\}$ of X and $f_i \in \mathcal{A}(U_i)$ for all i such that $\alpha(f_i) = \rho^X_{U_i}(g)$. Then

$$\begin{aligned}\alpha(\rho^{U_i}_{U_i\cap U_j}(f_i)) &= \rho^{U_i}_{U_i\cap U_j}(\alpha(f_i))\\ &= \rho^{U_i}_{U_i\cap U_j}(\rho^X_{U_i}(g))\\ &= \rho^X_{U_i\cap U_j}(g)\end{aligned}$$

The symmetry of $\rho^X_{U_i\cap U_j}(g)$ in i and j then implies that $\alpha(\rho^{U_i}_{U_i\cap U_j}(f_i)) = \alpha(\rho^{U_j}_{U_i\cap U_j}(f_j))$, which means that $\rho^{U_i}_{U_i\cap U_j}(f_i) = \rho^{U_j}_{U_i\cap U_j}(f_j)$ by the injectiveness of α. The sheaf condition then implies that there is a unique $f \in \mathcal{A}(X)$ such that $\rho^X_{U_i}(f) = f_i$.

To finish the proof, we now need to show that $\alpha(f) = g$. On every U_i, we have

$$\begin{aligned}\rho^X_{U_i}(\alpha(f)) &= \alpha(\rho^X_{U_i}(f))\\ &= \alpha(f_i)\\ &= \rho^X_{U_i}(g)\end{aligned}$$

so the uniqueness of the sheaf condition for $\mathcal{B}$ implies that $\alpha(f) = g$.□

Example: Since $0 \to$ [locally constant functions] $\to C^\infty \to \Omega^1 \to 0$ is exact, we know by the above theorem that $0 \to$ [locally closed functions]$(X) \to C^\infty(X) \to \Omega^1(X)$ is also exact. Note that the statement about exactness of sheaves can be checked on open discs in $\mathbb{R}^n$ (since any smooth manifold is locally diffeomorphic to an open disc in $\mathbb{R}^n$), yet the conclusion is valid on any smooth manifold.

Next time, we'll learn about sheaf cohomology, which extends $0 \to \mathcal{A}(X) \to \mathcal{B}(X) \to \mathcal{C}(X)$ to an infinite long exact sequence.

4. Appendix: Abelian categories by David Speyer

In class I defined "abelian category" as "a category where standard arguments about abelian groups are valid". If you are curious, here is the actual definition. This material is optional.

In this appendix, "sheaf" is short for "sheaf of abelian groups". Also, note that $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$, in this section, is the set of all sheaf maps from $\mathcal{A}$ to $\mathcal{B}$. There is an object which you may have heard of called the ***Hom-sheaf***, usually denoted $\mathcal{H}om$; that's not what I'm talking about.

The defining properties of an abelian category:

(1) Let $\mathcal{A}$ and $\mathcal{B}$ be two sheaves; let f and g be two maps $\mathcal{A} \to \mathcal{B}$. Then there is a map $f + g : \mathcal{A} \to \mathcal{B}$. This operation makes $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ into an abelian group. We have left and right distributivity: $f \circ (g + h) = f \circ g + f \circ h$ and $(g + h) \circ f = g \circ f + h \circ f$, where we have three sheaves $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, and f, g and h are maps between them such that these equations make sense.
(2) For any sheaf $\mathcal{A}$, we have $\mathrm{Hom}(\mathcal{A}, 0) \cong \mathrm{Hom}(0, \mathcal{A}) \cong \{0\}$ (the one element group).
(3) Given two sheaves $\mathcal{A}$ and $\mathcal{B}$, there is a sheaf $\mathcal{A}\oplus\mathcal{B}$ such that $\mathrm{Hom}(\mathcal{A}\oplus\mathcal{B}, \mathcal{C}) \cong \mathrm{Hom}(\mathcal{A}, \mathcal{C}) \oplus \mathcal{H}om(\mathcal{B}, \mathcal{C})$ and $\mathrm{Hom}(\mathcal{C}, \mathcal{A} \oplus \mathcal{B}) \cong \mathrm{Hom}(\mathcal{C}, \mathcal{A}) \oplus \mathcal{H}om(\mathcal{C}, \mathcal{B})$. These isomorphisms are compatible with composition in obvious ways.
(4) Let $f : \mathcal{A} \to \mathcal{B}$ be a map of sheaves. The composition $\mathcal{K}er(f) \to \mathcal{A} \to \mathcal{B}$ is zero. If $\mathcal{X} \to \mathcal{A}$ is a map of sheaves such that the composition $\mathcal{X} \to \mathcal{A} \to \mathcal{B}$ is zero, then there is a unique map of sheaves $\mathcal{X} \to \mathcal{K}er(f)$ such that the composition $\mathcal{X} \to \mathcal{K}er(f) \to \mathcal{A}$ is equal to the original map $\mathcal{X} \to \mathcal{A}$. The dual[1] assertions are true for $\mathcal{C}o\mathcal{K}er$.

[1]By "dual", I mean to reverse the direction of all arrows

(5) For any $f : \mathcal{A} \to \mathcal{B}$, we have $\mathcal{I}m(f) \cong \mathcal{C}o\mathcal{K}er(\mathcal{K}er(f) \to \mathcal{A}) \cong \mathcal{K}er(\mathcal{B} \to \mathcal{C}o\mathcal{K}er(f))$.

In a general abelian category, the phrasing of axioms 4 and 5 is that there exist objects with these properties; these objects are then defined to be the kernel, cokernel and image of f.

4.1. **Some weakenings of the abelian axioms.** Categories which satisfy the first axiom alone are sometimes called ***preadditive***, or ***enriched in abelian groups***. A good example of a preadditive category is "real vector spaces of dimension ≤ 10", because direct sum takes you outside the category. Any preadditive category has a canonical minimal enlargement to a category obeying the first three axioms. For example, "real vector spaces of dimension ≤ 10" embeds into "finite dimensional vector spaces". Perhaps for this reason, preadditive categories are rarely studied.

Categories which satisfy the first three axioms are called ***additive***. A good example of an additive category which does not satisfy axiom 4 is free modules over some ring (say $k[x, y]$), since the kernel or cokernel of a map of free modules need not be free[2].

Categories which satisfy the first four axioms are called ***preabelian***. A good example of a category which is preabelian but not abelian is Hausdorff topological groups. Consider the inclusion $\mathbb{Q} \to \mathbb{R}$, with the standard topologies. The zero group is both the kernel and the cokernel of this map. Then $\text{CoKer}(0 \to \mathbb{Q}) \cong \mathbb{Q}$ and $\text{Ker}(\mathbb{R} \to 0) \cong \mathbb{R}$, but $\mathbb{Q} \ncong \mathbb{R}$.

4.2. **Some important consequences of the axioms.** Given a map $f : \mathcal{A} \to \mathcal{B}$, the following are equivalent: The map f is a monomorphism (meaning that, for any $\mathcal{B} \to \mathcal{C}$, the induced map $\text{Hom}(\mathcal{A}, \mathcal{B}) \to \text{Hom}(\mathcal{A}, \mathcal{C})$ is an injection); the kernel $\mathcal{K}er(f)$ is isomorphic to 0; the sequence $0 \to \mathcal{A} \to \mathcal{B}$ is exact; the map f is injective. In a general abelian category, this is the definition of "injective". One defines "surjective" in a dual manner.

Given $\mathcal{A} \overset{\alpha}{\to} \mathcal{B} \overset{\beta}{\to} \mathcal{C}$ with $\alpha\beta = 0$, the following are equivalent: The natural map $\mathcal{I}m(f) \to \mathcal{K}er(\beta)$ is an isomorphism, the natural map $\mathcal{C}o\mathcal{K}er(\alpha) \to \mathcal{I}m(\beta)$ is an isomorphism, the sequence is exact. In a general abelian category, this is the definition of "exact".

Given a map $f : \mathcal{A} \to \mathcal{B}$, the sequences $0 \to \mathcal{K}er(f) \to \mathcal{A} \to \mathcal{I}m(f) \to 0$ and $0 \to \mathcal{I}m(f) \to \mathcal{B} \to \mathcal{C}o\mathcal{K}er(f) \to 0$ are exact.

A map $f : \mathcal{A} \to \mathcal{B}$ is an isomorphism if and only if it is injective and surjective.

Given a complex $\mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C}$, the natural map $\mathcal{C}o\mathcal{K}er(\mathcal{I}m(\alpha) \to \mathcal{K}er(\beta)) \to \mathcal{K}er(\mathcal{C}o\mathcal{K}er(\alpha) \to \mathcal{I}m(\beta))$ is an isomorphism. The sheaf on the left hand side of this map is, by definition, the ***cohomology of*** $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$. This is not the same as sheaf cohomology, which is a group, not a sheaf. I encourage you to forget this notion until we need it several months from now.

[2]More precisely, one must check that there is no free module which satisfies the conditions in axiom 4. The obvious object to try would be the kernel/cokernel of the map, but one must also check that nonobvious choices don't work.