NOTES FOR JANUARY 18

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Recall that, given a short exact sequence of sheaves on X

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0,$$

we have the long exact sequence

$$0 \longrightarrow \mathcal{A}(X) \longrightarrow \mathcal{B}(X) \longrightarrow \mathcal{C}(X).$$

Sheaf cohomology will extend this to an infinite long exact sequence,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{A}(X) & \longrightarrow & \mathcal{B}(X) & \longrightarrow & \mathcal{C}(X) & \longrightarrow \\ & & H^1(X,\mathcal{A}) & \longrightarrow & H^1(X,\mathcal{B}) & \longrightarrow & H^1(X,\mathcal{C}) & \longrightarrow \\ & & H^2(X,\mathcal{A}) & \longrightarrow & \dots & & & \end{array}$$

Today, we discuss how to define sheaf cohomology. I will prove as much as I can today, then pretend that everything has been proved from now on.

1. Some homological algebra lemmas

Homological Algebra Lemma 1: Given a diagram

$$\begin{array}{ccccc} A^1 & \xrightarrow{\partial} & A^2 & \xrightarrow{\partial} & A^3 \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ B^1 & \xrightarrow{\partial} & B^2 & \xrightarrow{\partial} & B^3 \end{array}$$

with $\partial\varphi = \varphi\partial$, there is an induced map $H^2(A^\bullet) \longrightarrow H^2(B^\bullet)$.

Construction: Given $x \in A^2$ with $\partial x = 0$, send $[x] \in H^2(A^\bullet)$ to $[\varphi(x)]$.

Note that $\partial\varphi(x) = \varphi\partial(x) = \varphi(0) = 0$, so the image lands in $\mathrm{Ker}(\partial)$. Also, if $x' = x + \partial y$ for some $y \in A^1$, then $\varphi(x') = \varphi(x) + \varphi\partial y = \varphi(x) + \partial\varphi(y)$ so $\varphi(x') - \varphi(x) \in \mathrm{Im}(\partial)$.

Therefore, $H^2(A^\bullet) \longrightarrow H^2(B^\bullet)$ is well-defined. $\square$

Homological Algebra Lemma 2: Given a diagram

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\partial} & A^{i-1} & \xrightarrow{\partial} & A^i & \xrightarrow{\partial} & A^{i+1} & \xrightarrow{\partial} & \cdots \\ & & \downarrow{\scriptstyle\varphi} & \swarrow{\scriptstyle S} & \downarrow{\scriptstyle\varphi} & \swarrow{\scriptstyle S} & \downarrow{\scriptstyle\varphi} & & \\ \cdots & \xrightarrow{\partial} & B^{i-1} & \xrightarrow{\partial} & B^i & \xrightarrow{\partial} & B^{i+1} & \xrightarrow{\partial} & \cdots \end{array}$$

such that $\partial S - S\partial = \varphi$, the map $\varphi_* : H^i(A^\bullet) \longrightarrow H^i(B^\bullet)$ is zero.

Proof: If $x \in A^i$ with $\partial x = 0$, then

$$\varphi(x) = \partial S(x) - S\partial(x) = \partial S(x)$$

so, $[\varphi(x)] \in H^i(B^\bullet)$ is zero. $\square$

Homological Algebra Lemma 3: Given

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{\partial} & A^{i-1} & \xrightarrow{\partial} & A^{i} & \xrightarrow{\partial} & A^{i+1} & \xrightarrow{\partial} & \cdots \\
 & & \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha & & \\
\cdots & \xrightarrow{\partial} & B^{i-1} & \xrightarrow{\partial} & B^{i} & \xrightarrow{\partial} & B^{i+1} & \xrightarrow{\partial} & \cdots \\
 & & \downarrow \beta & & \downarrow \beta & & \downarrow \beta & & \\
\cdots & \xrightarrow{\partial} & C^{i-1} & \xrightarrow{\partial} & C^{i} & \xrightarrow{\partial} & C^{i+1} & \xrightarrow{\partial} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the rows are complexes, the columns are short exact sequences, and all squares commute. Then, there exists δ making

$$H^{i-1}(A^\bullet) \xrightarrow{\alpha_*} H^{i-1}(B^\bullet) \xrightarrow{\beta_*} H^{i-1}(C^\bullet) \xrightarrow{\delta} H^{i}(A^\bullet) \xrightarrow{\alpha_*} H^{i}(B^\bullet) \xrightarrow{\beta_*} H^{i}(C^\bullet)$$

into a long exact sequence.

Construction: For $[z] \in H^{i-1}(C^\bullet)$, lift it to $z \in C^{i-1}$ with $\partial z = 0$. Lift to $y \in B^{i-1}$ s.t. $\beta y = z$. Now, $\partial \beta y = \partial z = 0$, so $\beta(\partial y) = 0$. Lift to $x \in A^i$ with $\alpha x = \partial y$. Set $\delta : [z] \mapsto [x]$. That this map is well defined, and that it makes a long exact sequence, is left for the problem sets.

2. Acyclic resolutions: the motivation for sheaf cohomology

Suppose we had an exact sequence of sheaves

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{C}^0 \longrightarrow \mathcal{C}^1 \longrightarrow \mathcal{C}^2 \longrightarrow \dots \qquad (*)$$

where $H^q(\mathcal{C}^p) = 0$ for all $q > 0$, all p.

Let the image of $\mathcal{C}^{k-1} \longrightarrow \mathcal{C}^k$ be $\mathcal{Z}^k$. So, we have short exact sequences

$$\begin{gathered}
0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{C}^0 \longrightarrow \mathcal{Z}^1 \longrightarrow 0 \\
0 \longrightarrow \mathcal{Z}^1 \longrightarrow \mathcal{C}^1 \longrightarrow \mathcal{Z}^2 \longrightarrow 0 \\
0 \longrightarrow \mathcal{Z}^2 \longrightarrow \mathcal{C}^2 \longrightarrow \mathcal{Z}^3 \longrightarrow 0
\end{gathered}$$

These would induce the long exact sequences

$$\begin{array}{cccc}
H^{k-1}(\mathcal{C}^0) \longrightarrow & H^{k-1}(\mathcal{Z}^1) \longrightarrow & H^k(\mathcal{E}) \longrightarrow & H^k(\mathcal{C}^0) \\
H^{k-2}(\mathcal{C}^1) \longrightarrow & H^{k-2}(\mathcal{Z}^2) \longrightarrow & H^{k-1}(\mathcal{Z}^1) \longrightarrow & H^{k-1}(\mathcal{C}^1) \\
\vdots & \vdots & \vdots & \vdots \\
H^0(\mathcal{C}^{k-1}) \longrightarrow & H^0(\mathcal{Z}^k) \longrightarrow & H^1(\mathcal{Z}^{k-1}) \longrightarrow & H^1(\mathcal{C}^{k-1})
\end{array}$$

From the maps we just wrote, and the assumption that $H^k(\mathcal{C}^j) = 0$, we get

$$H^k(\mathcal{E}) \cong H^{k-1}(\mathcal{Z}^1) \cong H^{k-2}(\mathcal{Z}^2) \cong \ldots \cong H^1(\mathcal{Z}^{k-1}) \cong \mathcal{Z}^k(X)/\operatorname{Im}(\mathcal{C}^{k-1}(X))$$

So we can express $H^k(\mathcal{E})$ in terms of global sections of various sheaves, if we have a resolution like $(*)$. This is great progress but, before continuing, we pause to simplify the expression $\mathcal{Z}^k(X)/\operatorname{Im}(\mathcal{C}^{k-1}(X))$.

Note that we have the exact sequences:

$$\begin{gathered}
0 \longrightarrow H^0(\mathcal{Z}^k) \longrightarrow H^0(\mathcal{C}^k) \longrightarrow H^0(\mathcal{Z}^{k+1}) \\
\text{and } 0 \longrightarrow H^0(\mathcal{Z}^{k+1}) \longrightarrow H^0(\mathcal{C}^{k+1})
\end{gathered}$$

So,

$$H^0(\mathcal{Z}^k) \cong \operatorname{Ker}(H^0(\mathcal{C}^k) \longrightarrow H^0(\mathcal{Z}^{k+1})) \cong \operatorname{Ker}(H^0(\mathcal{C}^k) \longrightarrow H^0(\mathcal{C}^{k+1}))$$

where the second equality is because $H^0(\mathcal{Z}^{k+1}) \longrightarrow H^0(\mathcal{C}^{k+1})$ is injective. This is our simpler formula for the "numerator" $\mathcal{Z}^k(X)$.

For the "denominator", $\mathrm{Im}(H^0(\mathcal{C}^{k-1}) \longrightarrow H^0(\mathcal{Z}^k))$, use injectivity to note that

$$\mathrm{Im}(H^0(\mathcal{C}^{k-1}) \longrightarrow H^0(\mathcal{Z}^k)) \cong \mathrm{Im}(H^0(\mathcal{C}^{k-1}) \longrightarrow H^0(\mathcal{C}^k))$$

Conclusion:

$$H^k(\mathcal{E}) \cong \frac{\mathrm{Ker}(\mathcal{C}^k(X) \longrightarrow \mathcal{C}^{k+1}(X))}{\mathrm{Im}(\mathcal{C}^{k-1}(X) \longrightarrow \mathcal{C}^k(X))} \quad (**)$$

So, if I know enough sheaves with vanishing higher cohomology, they will determine all other cohomology groups by the formula $(**)$

More generally, if I have any exact complex

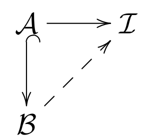
$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{C}^\bullet,$$

even without the hypothesis that $H^k(\mathcal{C}^j) = 0$, I get a map

$$\frac{\mathrm{Ker}(\mathcal{C}^k \longrightarrow \mathcal{C}^{k+1})}{\mathrm{Im}(\mathcal{C}^{k-1} \longrightarrow \mathcal{C}^k)} \longrightarrow H^k(\mathcal{E})$$

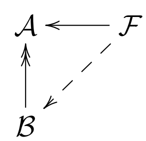
3. Injective sheaves

A sheaf $\mathcal{I}$ is called ***injective*** if, whenever we have the maps drawn in solid arrows



with $\mathcal{A} \hookrightarrow \mathcal{B}$ injective, there exists a map $\mathcal{B} \longrightarrow \mathcal{I}$ (drawn dashed) such that the diagram commutes.

The dual notion is ***projectivity***: a sheaf $\mathcal{F}$ is said to be ***projective*** if, for any diagram



with $\mathcal{B} \twoheadrightarrow \mathcal{A}$ surjective, then there exists a map $\mathcal{F} \longrightarrow \mathcal{B}$ s.t. the diagram commutes. In the category of modules, examples of projective objects are easy to come by: Any free module is projective. Examples of injective objects, either in modules or sheaves, tend to be rather bizarre. Nonetheless, they will be crucial for defining sheaf cohomology.

Key Lemma: For any sheaf $\mathcal{A}$ of abelian groups, there is an injective sheaf $\mathcal{I}$ of abelian groups and an injection $\mathcal{A} \hookrightarrow \mathcal{I}$.

Moreover, if $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$-modules, we can take $\mathcal{I}$ a sheaf of $\mathcal{O}_X$-modules, injective in the category of sheaves of $\mathcal{O}_X$-modules.

Given some sheaf $\mathcal{E}$ of abelian groups, using the lemma, we have

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{F}^1 \longrightarrow 0$$

where we define $\mathcal{F}^1$ as the cokernel. Continuing in this vein, we can find

$$0 \longrightarrow \mathcal{F}^1 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{F}^2 \longrightarrow 0$$

$$0 \longrightarrow \mathcal{F}^2 \longrightarrow \mathcal{I}^2 \longrightarrow \mathcal{F}^3 \longrightarrow 0$$

and so on. Then

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \dots$$

is exact.

We define

$$H^k(\mathcal{E}) = \frac{\mathrm{Ker}(\mathcal{I}^k(X) \longrightarrow \mathcal{I}^{k+1}(X))}{\mathrm{Im}(\mathcal{I}^{k-1}(X) \longrightarrow \mathcal{I}^k(X))}$$

In other words, we are imposing that injective sheaves have no higher cohomology, and then defining cohomology by $(**)$.

We must answer the following questions:

(1) Why is $H^k(\mathcal{E})$ independent of the choice of $\mathcal{I}^\bullet$?
(2) Given a map of sheaves $\mathcal{A} \mapsto \mathcal{B}$, what is the map $H^k(\mathcal{A}) \longrightarrow H^k(\mathcal{B})$?
(3) How do you build the long exact sequence?

Provisionally, I will write $H^k(\mathcal{E}, \mathcal{I}^\bullet)$ until I prove independence from $\mathcal{I}^\bullet$.

Claim 1: Given

$$\begin{CD} 0 @>>> \mathcal{A} @>>> \mathcal{I}^0 @>>> \mathcal{I}^1 @>>> \mathcal{I}^2 @>>> \cdots \\ @. @VVV \\ 0 @>>> \mathcal{B} @>>> \mathcal{J}^0 @>>> \mathcal{J}^1 @>>> \mathcal{J}^2 @>>> \cdots \end{CD}$$

I can extend to vertical maps such that all the squares commute.

Assuming this claim, we obtain

$$\begin{CD} 0 @>>> \mathcal{A}(X) @>>> \mathcal{I}^0(X) @>>> \mathcal{I}^1(X) @>>> \mathcal{I}^2(X) @>>> \cdots \\ @. @VVV @VVV @VVV @VVV \\ 0 @>>> \mathcal{B}(X) @>>> \mathcal{J}^0(X) @>>> \mathcal{J}^1(X) @>>> \mathcal{J}^2(X) @>>> \cdots \end{CD} \qquad (***)$$

By homological algebra lemma 1, we get a map

$$H^k(\mathcal{A}, \mathcal{I}^\bullet) \longrightarrow H^k(\mathcal{B}, \mathcal{J}^\bullet).$$

So this gives us a map $H^k(\mathcal{A}, \mathcal{I}^\bullet) \longrightarrow H^k(\mathcal{B}, \mathcal{J}^\bullet)$, partially answering question (2). However, we have not shown that this map is independent of the choices made in constructing the vertical maps in $(***)$. We will now show this independence.

Claim 2: Suppose we have $\mathcal{A}$, $\mathcal{B}$, $\mathcal{I}$, $\mathcal{J}$ as above and φ, ψ two sets of vertical maps such that squares commute

$$\begin{CD} \mathcal{I}^0 @>\partial>> \mathcal{I}^1 @>\partial>> \mathcal{I}^2 @>\partial>> \cdots \\ @V{\varphi}V{\psi}V @V{\varphi}V{\psi}V @V{\varphi}V{\psi}V \\ \mathcal{J}^0 @>\partial>> \mathcal{J}^1 @>\partial>> \mathcal{J}^2 @>\partial>> \cdots \end{CD}$$

Then there is a map $S : \mathcal{I}^k \longrightarrow \mathcal{J}^{k-1}$ s.t. $\partial S - S\partial = \varphi - \psi$.

So, by second homological algebra lemma

$$\varphi^* - \psi^* : H^k(\mathcal{A}) \longrightarrow H^k(\mathcal{B})$$

is zero, so $\varphi^* = \psi^*$ on $H^k(\mathcal{A})$. We see that if φ and ψ are two ways of completing the diagram in $(***)$, then they induce the same map on cohomology. This resolves question (2).

Now, let's go back to question (1). Suppose we have two resolutions:

$$\begin{CD} 0 @>>> \mathcal{A} @>>> \mathcal{I}^0 @>>> \mathcal{I}^1 @>>> \mathcal{I}^2 @>>> \cdots \\ @. @| \\ 0 @>>> \mathcal{A} @>>> \mathcal{J}^0 @>>> \mathcal{J}^1 @>>> \mathcal{J}^2 @>>> \cdots \end{CD}$$

As above, fill in the diagram

$$\begin{CD} 0 @>>> \mathcal{A} @>>> \mathcal{I}^0 @>>> \mathcal{I}^1 @>>> \mathcal{I}^2 @>>> \cdots \\ @. @| @VVV @VVV @VVV \\ 0 @>>> \mathcal{A} @>>> \mathcal{J}^0 @>>> \mathcal{J}^1 @>>> \mathcal{J}^2 @>>> \cdots \\ @. @| @VVV @VVV @VVV \\ 0 @>>> \mathcal{A} @>>> \mathcal{I}^0 @>>> \mathcal{I}^1 @>>> \mathcal{I}^2 @>>> \cdots \end{CD}$$

We get maps $\alpha_{IJ} : H^k(\mathcal{A}, \mathcal{I}^\bullet) \longrightarrow H^k(\mathcal{A}, \mathcal{J}^\bullet)$ and $\alpha_{JI} : H^k(\mathcal{A}, \mathcal{J}^\bullet) \longrightarrow H^k(\mathcal{A}, \mathcal{I}^\bullet)$.

Now $\alpha_{IJ} \circ \alpha_{JI}$ is induced by taking the composite vertical maps in the above diagram. But we have proved that this induced map is independent of which vertical maps we use. So $\alpha_{IJ} \circ \alpha_{JI}$ must be the same map from $H^k(\mathcal{A}, \mathcal{I}^\bullet)$ to itself as would be induced by just using the identity map $\mathcal{I}^K \to \mathcal{I}^k$ in every column, In other words, $\alpha_{IJ} \circ \alpha_{JI}$ is the identity. Similarly, $\alpha_{JI} \circ \alpha_{IJ} = Id$.

So, I have given you an isomorphism $H^k(\mathcal{A}, \mathcal{I}^\bullet) \cong H^k(\mathcal{A}, \mathcal{J}^\bullet)$. This resolves question (1).

Remark not made in class: Of course, to be careful, you should check that this isomorphism is natural, meaning that it is compatible with the maps constructed in our answer to question (2). I won't do this.

Now, for question (3).

Claim 3: Suppose we have

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{I}^0 & \longrightarrow & \mathcal{I}^1 & \longrightarrow & \cdots \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & \mathcal{B} & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{J}^0 & \longrightarrow & \mathcal{J}^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Then we can extend to

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{I}^0 & \longrightarrow & \mathcal{I}^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{B} & \dashrightarrow & \mathcal{I}^0 \oplus \mathcal{J}^0 & \dashrightarrow & \mathcal{I}^1 \oplus \mathcal{J}^1 & \dashrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{J}^0 & \longrightarrow & \mathcal{J}^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the vertical maps are the obvious ones.

Assuming this claim, we get

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
\mathcal{I}^0(X) & \longrightarrow & \mathcal{I}^1(X) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \\
\mathcal{I}^0(X) \oplus \mathcal{J}^0(X) & \longrightarrow & \mathcal{I}^1(X) \oplus \mathcal{J}^1(X) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \\
\mathcal{J}^0(X) & \longrightarrow & \mathcal{J}^1(X) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

Moreover, the columns are exact because they are just split sequences. So the third homological algebra lemma provides the long exact sequence, answering question (3).

We have now answered all of our questions, except for proving the claims. For this I refer you to any book on homological algebra.