NOTES FOR JANUARY 20

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On Tuesday, we saw the following result: if we have an exact sequence

$$0 \to \mathcal{E} \to \mathcal{C}^0 \to \mathcal{C}^1 \to \dots$$

of sheaves on a space X, and we know that

$$H^q(\mathcal{C}^k) = 0$$

for all k and for $q > 0$, then the cohomology of $\mathcal{E}$ is the cohomology of the complex

$$0 \to \mathcal{C}^0(X) \to \mathcal{C}^1(X) \to \mathcal{C}^2(X) \to \dots$$

This observation allowed us to define cohomology by declaring that the higher cohomology groups should vanish for injective sheaves. Injective sheaves are good for making definitions in sheaf cohomology, and for proving basic properties, but we don't want to think about them after that, so we need to think of other ways to make sheaf cohomology vanish. To that end, in this lecture we will discuss sheaves on ringed spaces with partitions of unity and Čech cohomology. We will conclude by proving de Rham's theorem.

1. Partitions of unity and vanishing of sheaf cohomology

Our underlying space X will always be assumed ***paracompact*** today, which means that every open cover has a locally finite refinement. Here, ***refinement*** is defined as follows: given open covers $\{U_i\}$ and $\{V_j\}$, we say $\{V_j\}$ refines $\{U_i\}$ if for all j there is an i with $V_j \subseteq U_i$. A cover is ***locally finite*** if every point has a neighborhood which meets only finitely many open sets in the cover.

Given a paracompact space X with a sheaf of commutative rings $\mathcal{O}$ on it, we say that $\mathcal{O}$ "has ***partitions of unity***" if, for any locally finite cover U_i, there exist global sections

$$f_i \in \mathcal{O}(X)$$

and open sets V_i such that

- $U_i \cup V_i = X$
- $\rho^X_{V_i}(f_i) = 0$
- $\sum f_i = 1$

To understand the (a priori infinite) sum in the last condition, notice that for every $x \in X$ there is a neighborhood $W \ni x$ such that only finitely many of the U_i meet W. Then for every U_i not meeting W, we have $W \subseteq V_i$ so that the restriction

$$\rho^X_W(f_i) = 0.$$

We thus may compute the sum by taking

$$\sum \rho^X_W(f_i),$$

since there are only finitely many nonzero terms in this sum. Thus, the third condition says that for all $x \in X$ there is a $W \ni x$, meeting only finitely many U_i, such that

$$\sum \rho^X_W(f_i) = 1.$$

***Remark* 1.1.** — Instead of introducing the V_i, why do we not just ask that $f_i|_{X\setminus U_i} = 0$? The reason is that for general sheaves $\mathcal{O}$, restriction to a closed set doesn't make sense. For sheaves of functions on topological spaces, however, it *does* makes sense and it turns out that if the space is T_3 we'll get an equivalent definition.

A fundamental fact from differential geometry, which we'll use later, is that the sheaf of C^∞ functions on a manifold has partitions of unity.

Proposition 1.2. *If X is paracompact, $\mathcal{O}$ is a sheaf of commutative rings with partitions of unity, and $\mathcal{E}$ is any sheaf of $\mathcal{O}$-modules, then*

$$H^q(\mathcal{E}) = 0$$

for all $q > 0$.

Proof. We'll induct on q. The base case $q = 1$ is the hardest part of the proof. Find an injective sheaf of $\mathcal{O}$-modules $\mathcal{I}$ with

$$0 \to \mathcal{E} \to \mathcal{I} \xrightarrow{\beta} \mathcal{F} \to 0$$

(here we are defining $\mathcal{F}$ to be this cokernel - notice that $\mathcal{F}$ has the structure of a sheaf of $\mathcal{O}_X$-modules and not just a sheaf of abelian groups).

From the long exact sequence in cohomology, we get an exact sequence

$$0 \to H^0(\mathcal{E}) \to H^0(\mathcal{I}) \to H^0(\mathcal{F}) \to H^1(\mathcal{E}) \to 0, \tag{1}$$

and then for each $q > 0$ we get the exact sequence

$$0 \to H^q(\mathcal{F}) \to H^{q+1}(\mathcal{E}) \to 0.$$

By induction we are reduced to the case where $q = 1$. If we can show that

$$H^0(\mathcal{I}) \to H^0(\mathcal{F}) \to 0$$

then the exact sequence (1) will finish the argument.

Let $a \in \mathcal{F}(X) = H^0(\mathcal{F})$. Since β is a surjective map of sheaves, there is an open cover $\{U_i\}$ of X and there are sections $b_i \in \mathcal{I}(U_i)$ such that $\beta(b_i) = a|_{U_i}$. Refining the cover if necessary, we may assume it is locally finite. Find V_i and f_i as in the definition of partitions of unity. We'll define $c_i \in \mathcal{I}(X)$ by setting

- $c_i|_{U_i} = f_i b_i$,
- $c_i|_{V_i} = 0$.

These local sections really do glue to give global sections c_i, because they are consistent on overlaps. We check this:

$$f_i b_i|_{U_i \cap V_i} = f_i|_{U_i \cap V_i} b_i|_{U_i \cap V_i} = 0 \cdot b_i|_{U_i \cap V_i} = 0.$$

We claim that

$$\beta\left(\sum c_i\right) = a,$$

where we define the infinite sum as before (we only need to check this claim on an open cover). For $x \in X$, find an open $W \ni x$ such that all but finitely many U_i do not meet W. Then

$$\beta\left(\sum c_i\right)|_W = \sum \beta(c_i)|_W.$$

If U_i does not meet W, then $W \subseteq V_i$ and so $c_i|_W = 0$. Thus the latter sum is just

$$\sum_{U_i \cap W \neq \emptyset} \beta(c_i)|_W = \sum_{U_i \cap W \neq \emptyset} f_i \cdot \beta(b_i)|_W = \sum_{U_i \cap W \neq \emptyset} f_i \cdot a|_W = \sum f_i \cdot a|_W = 1 \cdot a|_W = a|_W,$$

which finishes the proof. □

2. Čech cohomology

Now we will talk about Čech complexes. These will allow us to compute the cohomology of a sheaf given the data of an open cover on which we know the cohomology vanishes. More precisely, for $\mathcal{E}$ a sheaf on X, and U an open set, we write $\mathcal{E}_U$ for the sheaf on X defined by

$$\mathcal{E}_U(V) = \mathcal{E}(U \cap V).$$

There is a map of sheaves $\mathcal{E} \to \mathcal{E}_U$ given on an open V by $\rho^V_{U\cap V}$. Now let $\{U_i\}$ be a locally finite open cover of X. We define the ***Čech complex*** of sheaves on X:

$$0 \to \mathcal{E} \to \bigoplus \mathcal{E}_{U_i} \to \bigoplus_{i<j} \mathcal{E}_{U_i \cap U_j} \to \bigoplus_{i<j<k} \mathcal{E}_{U_i \cap U_j \cap U_k} \to \dots \tag{2}$$

(Here the U_i are indexed by some ordered set I.) The maps in the complex are defined as follows:

$$\mathcal{E}_{U_1 \cap \ldots \cap \widehat{U_s} \cap \ldots \cap U_r} \to \mathcal{E}_{U_1 \cap U_2 \cap \ldots \cap U_r}$$

will be restriction followed by multiplication by $(-1)^{s-1}$. In fact, we claim that (2) is not only a complex, but an exact sequence of sheaves, which we will now check on stalks. For $x \in X$, let S be the stalk $\mathcal{E}_x$ and label those sets in our locally finite cover which contain x as

$$U_{i_1}, U_{i_2}, \ldots, U_{i_m} \ni x.$$

On the stalk S, our complex becomes

$$0 \to S \to S^{\oplus m} \to S^{\oplus \binom{m}{2}} \to \ldots \to S^{\oplus \binom{m}{m-1}} \to S \to 0 \to \ldots$$

This is the complex that computes the reduced simplicial cohomology of the m-simplex in topology (with coefficients in S). As simplices are contractible, the cohomology of this complex vanishes.

Now, if X has a locally finite cover $\{U_i\}$ such that

$$H^q(\mathcal{E}_{U_{i_1} \cap \ldots \cap U_{i_m}}) = 0 \tag{3}$$

for all $q > 0$, and all $i_1, \ldots, i_m$, $m > 0$, then we conclude from the exactness of (2) that $H^k(\mathcal{E})$ is the cohomology of the complex

$$0 \to \bigoplus \mathcal{E}_{U_i}(X) \to \bigoplus \mathcal{E}_{U_i \cap U_j}(X) \to \ldots$$

or in other words, the cohomology of the complex

$$0 \to \bigoplus \mathcal{E}(U_i) \to \bigoplus \mathcal{E}(U_i \cap U_j) \to \ldots \tag{4}$$

Remark 2.1. — Consider the cohomology $\check{H}^k(\mathcal{E}, U_i)$ of the complex (4) with respect to an arbitrary locally finite cover $\{U_i\}$, possibly not satisfying condition (3). If $\{U_i\}$ and $\{V_i\}$ are locally finite covers with $\{V_i\}$ refining $\{U_i\}$, then there is a natural map

$$\check{H}^k(\mathcal{E}, U_i) \to \check{H}^k(\mathcal{E}, V_i)$$

and Godement showed that for any sheaf of abelian groups $\mathcal{E}$ on a paracompact space,

$$H^k(\mathcal{E}) = \varinjlim_{U_i} \check{H}^k(\mathcal{E}, U_i).$$

Thus the Čech complex could be used to give a definition of sheaf cohomology without reference to injective resolutions. In fact, suppose we have a short exact sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

and a locally finite cover U_i such that

- $0 \to \mathcal{A}(U_{i_1} \cap \ldots \cap U_{i_k}) \to \mathcal{B}(U_{i_1} \cap \ldots \cap U_{i_k}) \to \mathcal{C}(U_{i_1} \cap \ldots \cap U_{i_k}) \to 0$ is exact for all $i_1, \ldots, i_k$.
- $H^q(\mathcal{A}_{U_{i_1} \cap \ldots \cap U_{i_k}}) = H^q(\mathcal{B}_{U_{i_1} \cap \ldots \cap U_{i_k}}) = H^q(\mathcal{C}_{U_{i_1} \cap \ldots \cap U_{i_k}}) = 0$ for $q > 0$ and all $i_1, \ldots, i_k$.

Then we get a short exact sequence of chain complexes:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus \mathcal{A}(U_i) & \longrightarrow & \bigoplus \mathcal{A}(U_i \cap U_j) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus \mathcal{B}(U_i) & \longrightarrow & \bigoplus \mathcal{B}(U_i \cap U_j) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus \mathcal{C}(U_i) & \longrightarrow & \bigoplus \mathcal{C}(U_i \cap U_j) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

The long exact sequence in sheaf cohomology follows from this, exactly as before.

3. de Rham's Theorem

Theorem 3.1 (de Rham)**.** *Let X be a smooth manifold of dimension n. Then the cohomology of the complex*

$$0 \to \Omega^0(X) \to \Omega^1(X) \to \Omega^2(X) \to \ldots \to \Omega^n(X) \to 0 \tag{5}$$

is isomorphic to the (topological) cohomology $H^*(X, \mathbb{R})$.

Proof. Letting $\mathcal{LC}$ be the sheaf of locally constant functions on X, we will show that both cohomology groups referenced in the theorem are isomorphic to the sheaf cohomology $H^*(X, \mathcal{LC})$. (This sheaf is usually denoted $\mathbb{R}$, but that makes the notation for cohomology ambiguous, until we prove the result.) Recall that C^∞ has partitions of unity. As Ω^k has the structure of a C^∞-module, we know that

$$H^q(X, \Omega^k) = 0.$$

We also know, by the Poincaré lemma, that we have an exact sequence of sheaves

$$0 \to \mathcal{LC} \to \Omega^0 \to \Omega^1 \to \Omega^2 \to \ldots \to \Omega^k \to 0.$$

Thus the cohomology of the de Rham sequence (5) computes $H^k(X, \mathcal{LC})$.

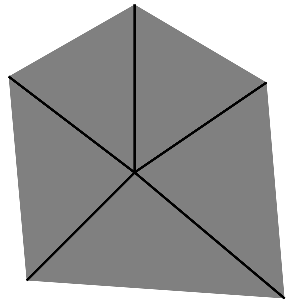
To finish, we need to show that the cohomology of the sheaf $\mathcal{LC}$ also computes the cohomology of X. We'll work with simplicial cohomology; take a triangulation of X. For v a vertex, and F a face containing v, we define a subset of F:

$$N(v, F) = \{w \in F | w \notin F', \text{ with } F' \text{ a face of } F \text{ not containing } v\}.$$

From these we can build an open cover $\{U_v\}$ of X by setting

$$U_v = \bigcup_{F \ni v} N(v, F).$$

In the illustration, we are looking at a two dimensional manifold and v is the vertex at the center. U_v contains the 5 triangles with v as a vertex, but does not contain the sides of them which are opposite v.



Given vertices $v_1, \ldots, v_r$, we observe that $U_{v_1} \cap U_{v_2} \cap \ldots \cap U_{v_r}$ is empty if $v_1, \ldots, v_r$ do not form a face and is contractible if $v_1, \ldots, v_r$ form a face.

In order to compute Čech cohomology with this cover, we thus need to know that

$$H^k(\mathcal{LC}_{U_{v_1} \cap \ldots \cap U_{v_r}})$$

vanishes for any choice of r vertices $v_1, \ldots, v_r$ which form a face. But this is the same as a sheaf cohomology group on a smaller space, namely the cohomology group

$$H^k(U_{v_1} \cap \ldots \cap U_{v_r}, \mathcal{LC}|_{U_{v_1} \cap \ldots \cap U_{v_r}}).$$

We have already seen that this, in turn, is the cohomology of

$$0 \to \Omega^0(U_{v_1} \cap \ldots \cap U_{v_r}) \to \Omega^1(U_{v_1} \cap \ldots \cap U_{v_r}) \to \Omega^2(U_{v_1} \cap \ldots \cap U_{v_r}) \to \ldots$$

which vanishes by the Poincaré lemma, due to the contractibility of $U_{v_1} \cap \ldots \cap U_{v_r}$. Thus we may use our cover and the Čech complex to compute $H^k(\mathcal{LC})$.

On empty intersections, the sections of $\mathcal{LC}$ are of course 0, and on contractible intersections the sections are isomorphic to $\mathbb{R}$. The Čech complex thus becomes

$$0 \to \bigoplus_{v_i} \mathbb{R} \to \bigoplus_{v_i, v_j \text{ an edge}} \mathbb{R} \to \bigoplus_{v_i, v_j, v_k \text{ a face}} \mathbb{R} \to \ldots$$

which is exactly the simplicial cochain complex for X. This implies that the sheaf cohomology of $\mathcal{LC}$ is isomorphic to the $\mathbb{R}$-valued cohomology of X, as desired. $\square$