# NOTES FOR 25 JANUARY 2011

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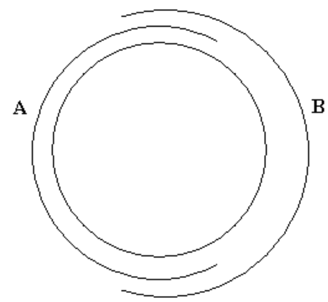
## 1. Sheaf Cohomology Examples

Basic point: Sheaf cohomology depends on the space and the sheaf. In basic topology, for $X$ a space, $A$ an abelian group, we have groups $H^k(X, A)$; the coefficients $A$ matter. For "reasonable" spaces (e.g. finite simplicial complexes),

$$H^k(X, \text{locally constant } A\text{-valued functions}) = H^k_{\text{top}}(X, A).$$

On $S^1$, look at the sheaf $\mathcal{LC}$ (sheaf of locally constant real-valued functions). Take the open cover $A \cup B$ illustrated below. To use Čech cohomology, we need to know higher cohomology vanishes on each intersection. Here, each intersection is contractible, hence higher cohomology vanishes.




Then since $A$ and $B$ are connected but $A \cap B$ has two connected components,

(1) $$0 \longrightarrow \mathcal{LC}(A) \oplus \mathcal{LC}(B) \longrightarrow \mathcal{LC}(A \cap B) \longrightarrow 0$$

$$0 \longrightarrow \mathbb{R} \oplus \mathbb{R} \xrightarrow{\ M\ } \mathbb{R}^{\oplus 2} \longrightarrow 0$$

where $M$ is the matrix

$$M = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

(our matrices act on the left, so we think of these as column vectors). Hence

$$H^0(S^1, \mathcal{LC}) = \ker M = \mathbb{R} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$H^1(S^1, \mathcal{LC}) = \operatorname{coker} M \cong \mathbb{R}.$$

For a larger example, we could think of $S^2$ as a cube, with an open cover consisting of open sets around each facet. We get the Čech complex

$$0 \to \mathbb{R}^6 \to \mathbb{R}^{12} \to \mathbb{R}^8 \to 0$$

with cohomology

$$H^0(S^2, \mathcal{LC}) = \mathbb{R}$$
$$H^1(S^2, \mathcal{LC}) = 0$$
$$H^2(S^2, \mathcal{LC}) = \mathbb{R}.$$

We will now work through the de Rham isomorphism for the circle (doing it for the sphere is a good exercise). On the circle, we have an exact sequence of sheaves

$$0 \to \mathcal{LC} \to C^\infty \to \Omega^1 \to 0$$

In general the third term would be $Z^1$ (closed 1-forms), but on a one dimensional manifold, every 1-form is closed. The long exact sequence is

$$0 \to \mathcal{LC}(S^1) \to C^\infty(S^1) \to \Omega^1(S^1) \to H^1(S^1, \mathcal{LC}) \to 0$$

where the last 0 is because the sheaf $C^\infty$ has partitions of unity, and hence higher cohomology vanishes.

We have the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{R} \oplus \mathbb{R} & \xrightarrow{\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}} & \mathbb{R}^{\oplus 2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^\infty(A) \oplus C^\infty(B) & \longrightarrow & C^\infty(A \cap B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1(A) \oplus \Omega^1(B) & \longrightarrow & \Omega^1(A \cap B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

where the first row is the exact sequence (1), and columns are exact by the Poincaré lemma.

Given $\omega \in \Omega^1(S^1)$, consider it as $(\omega_1, \omega_2) \in \Omega^1(A) \oplus \Omega^1(B)$ by restricting to each interval. From our diagram, we can lift this up to $(f_1, f_2) \in C^\infty(A) \oplus C^\infty(B)$ such that $df_i = \omega_i$; i.e.,

$$f_1(\theta) = \int_P^\theta \omega_1$$
$$f_2(\theta) = \int_P^\theta \omega_2$$

where $P$ is a point in one of the connected components of $A \cap B$, and the integrals are computed in $A$ and $B$ respectively (so in opposite directions). Next we go across to $f_1 - f_2 \in C^\infty(A \cap B)$. This is 0 in the connected component of $A \cap B$ containing $P$. On the other connected component, this is $\oint \omega$. Therefore, $f_1 - f_2$ is locally constant, with values 0 and $\oint \omega$ on the two connected components of $A \cap B$. Hence

$$f_1 - f_2 = \begin{pmatrix} \oint \omega \\ 0 \end{pmatrix}$$

in $\mathcal{LC}(A \cap B) \cong \mathbb{R}^2$. The image in $H^1(S^1, \mathcal{LC})$ is therefore $\oint \omega$.

There are two harder versions in HW 3: one involves using two complexes, $0 \to \mathcal{LC} \to C^\infty \to Z^1 \to 0$ and $0 \to Z^1 \to \Omega^1 \to \Omega^2 \to 0$.

## 2. Complex Differential Operators

The main objects today will be smooth functions $f : \mathbb{C}^n \to \mathbb{C}$,

$$(x_1 + iy_1, ..., x_n + iy_n) \mapsto (u, v)$$

where $u(x_1, ..., x_n, y_1, ..., y_n)$ and $v(x_1, ..., x_n, y_1, ..., y_n)$ are elements of $C^\infty(\mathbb{R}^{2n})$. For such an $f$,

$$df = du + idv.$$

In particular, $z_1, ..., z_n, \overline{z_1}, ..., \overline{z_n}$ are such functions $\mathbb{C}^n \to \mathbb{C}$, so we have $dz_1, ..., dz_n, d\overline{z_1}, ..., d\overline{z_n}$ with

$$dz_k = dx_k + idy_k \quad \text{and} \quad d\overline{z_k} = dx_k - idy_k.$$

We can rearrange to solve for $dx_k$ and $dy_k$:

$$dx_k = \frac{dz_k + d\overline{z_k}}{2} \qquad dy_k = \frac{dz_k - d\overline{z_k}}{2i}.$$

Geometrically, $dz_k$ is the 1-form that takes a tangent vector and returns the $k$th component (as a complex number); $d\overline{z_k}$ returns the complex conjugate. Every smooth 1-form can be uniquely written as

$$\sum f_k(z)dz_k + \sum g_k(z)d\overline{z_k}$$

where $f_1, ..., f_n, g_1, ..., g_n$ are smooth $\mathbb{C}$-valued functions on $\mathbb{C}^{2n}$.

A 1-form $\omega$ is called a ***$(1,0)$-form*** if it is of the form

$$\sum f_k(z)dz_k.$$

Similarly, $\omega$ is called a ***$(0,1)$-form*** if it is of the form

$$\sum g_k(z)d\overline{z_k}.$$

For $v$ a tangent vector to $\mathbb{C}^n$, let $Jv$ be $v$ rotated by $i$. Then $\omega$ is a $(1,0)$-form if and only if $\omega(Jv) = i\omega(v)$ and $\omega$ is a $(0,1)$-form if and only if $\omega(Jv) = -i\omega(v)$.

A smooth manifold $X$ with a map

$$J : T_*X \to T_*X$$

obeying $J^2 = -\mathrm{Id}$ and preserving the base (i.e., $J$ takes tangent vectors at $P$ to other tangent vectors at $P$) is called ***almost complex***. On an almost complex manifold, $(0,1)$-forms and $(1,0)$-forms are defined in terms of $J$ as above.

An almost complex manifold which locally looks like an open set in $\mathbb{C}^n$ is called a ***complex*** manifold.

(The main difference between complex and almost complex: on almost complex manifolds, we can't necessarily find $n$ holomorphic functions like $z_1, ..., z_n$ so that forms can be written in terms of $dz_i$ and $d\overline{z_i}$.)

For $f : \mathbb{C} \to \mathbb{C}$ mapping $(x, y)$ to $(u(x, y), v(x, y))$, we have

$$\begin{aligned}
df &= du + idv \\
&= \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + i\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right) \\
&= \frac{\partial u}{\partial x}\left(\frac{dz + d\overline{z}}{2}\right) + \frac{\partial u}{\partial y}\left(\frac{dz - d\overline{z}}{2i}\right) + i\frac{\partial v}{\partial x}\left(\frac{dz + d\overline{z}}{2}\right) + \frac{\partial v}{\partial y}\left(\frac{dz - d\overline{z}}{2}\right) \\
&= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial x}\right)dz + \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial x}\right)d\overline{z} \\
&= \frac{1}{2}\left(\frac{\partial(u+iv)}{\partial x} - i\frac{\partial(u+iv)}{\partial y}\right)dz + \frac{1}{2}\left(\frac{\partial(u+iv)}{\partial x} + i\frac{\partial(u+iv)}{\partial y}\right)d\overline{z} \qquad (2)
\end{aligned}$$

The 1-form $df$ is a $(1,0)$-form if and only if the $d\overline{z}$ part vanishes; i.e., if and only if

$$\begin{aligned}
\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= 0 \\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= 0.
\end{aligned}$$

These are precisely the Cauchy-Riemann equations. So $df$ is a $(1,0)$-form if and only if $f$ is holomorphic. Moreover, if $f$ is holomorphic, the above equation for $df$ simplifies to

$$df = f'(z)dz.$$

In general, if $f : \mathbb{C}^n \to \mathbb{C}$, $f$ is holomorphic if and only if $df$ is a $(1,0)$-form, in which case

$$df = \sum \frac{\partial f}{\partial z_i}dz_i.$$

Let $df = \sum g_k dz_k + \sum h_k d\overline{z_k}$. Then we define ***del*** and ***del-bar*** as

$$\begin{aligned}
\partial f &= \sum g_k dz_k \\
\overline{\partial} f &= \sum h_k d\overline{z_k}.
\end{aligned}$$

The automorphism $J$ on the complexification of the tangent bundle has eigenvalues $i$ and $-i$. $\partial$ is the projection onto the $i$ eigenspace and $\overline{\partial}$ is the projection on to the $-i$ eigenspace.

Given $\omega \in \Omega^k(X)$ with $k = p+q$, $\omega$ is called a $(p,q)$-form if

$$\omega = \sum f_{I,J} dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\overline{z_{j_1}} \wedge \ldots \wedge d\overline{z_{j_q}}.$$

Every smooth $k$-form is uniquely a sum of a $(k,0)$-form, a $(k-1,1)$-form, ..., and a $(0,k)$-form. On a $(p,q)$-form, we have

$$\begin{aligned} d\left(\sum f_{I,J} dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\overline{z_{j_1}} \wedge \ldots \wedge d\overline{z_{j_q}}\right) &= \sum (df_{I,J}) dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\overline{z_{j_1}} \wedge \ldots \wedge d\overline{z_{j_q}} \\ &= \sum \left(\sum \ldots dz_i + \sum \ldots d\overline{z_j}\right)\left(dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\overline{z_{j_1}} \wedge \ldots \wedge d\overline{z_{j_q}}\right), \end{aligned}$$

a sum of $(p+1,q)$-forms and $(p,q+1)$-forms. We define $\partial$ as the projection onto the $(p+1,q)$ part and $\overline{\partial}$ as the projection onto the $(p,q+1)$ part.

$\partial$ (and same for $\overline{\partial}$) obeys all the usual formal properties:

(1) $\partial(u+v) = \partial u + \partial v$
(2) $\partial(au) = a\partial u$
(3) $\partial(u \cdot v) = u\partial v + v\partial u$
(4) $\partial(\omega \wedge \eta) = (-1)^k \omega \wedge \partial\eta + \partial w \wedge \eta$ where $\omega \in \Omega^k$.

Additionally, if $f$ is analytic, $\overline{\partial}(f \cdot v) = f\overline{\partial}(v)$.

To integrate

$$\int_\gamma f(z) dz$$

for some $f : \mathbb{C} \to \mathbb{C}$, we break up $\gamma$ into $N$ subintervals and take the limit

$$\lim_{N\to\infty} \sum_{i=0}^{N} f(z_i) \cdot \overset{\text{1-form}}{(dz)} \overset{\text{tan. vector}}{(z_{i+1} - z_i)} = \lim_{N\to\infty} \sum_{i=0}^{N} f(z_i)(z_{i+1} - z_i)$$

to get the usual complex integral.

For $f$ analytic on some domain $D$,

$$\int_{\partial D} f dz = \int_D d(f dz) = \int_D df \wedge dz = \int_D \frac{\partial f}{\partial z} dz \wedge dz = \int_D 0 = 0.$$

For $f : \mathbb{C}^n \to \mathbb{C}$, we define $\frac{\partial f}{\partial z_i}$ and $\frac{\partial f}{\partial \overline{z_i}}$ such that

$$df = \sum \frac{\partial f}{\partial z_i} dz_i + \sum \frac{\partial f}{\partial \overline{z_i}} d\overline{z_i}.$$

Note that this is defined whenever $f$ is smooth (not necessarily holomorphic). Note that $f$ is holomorphic if and only if $\frac{\partial f}{\partial \overline{z_1}}, \ldots, \frac{\partial f}{\partial \overline{z_n}}$ are 0.

**Something I didn't say in class (but should have)** One has the explicit formulas:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

These can be read off from the last line in equation (2).

**A warning about the partial notation**: on $\mathbb{R}^2$, consider the coordinate charts $(x,y)$ and $(x, x+y)$ (a grid and a slanted grid). Let $f$ be the same function on each. Then $\frac{\partial f}{\partial x}$ means different things in the two charts! (Hence the partial notation can be misleading.)

Next time: The ***Dolbeault complex***[1]

$$0 \to \text{holomorphic functions} \to \Omega^{0,0} \xrightarrow{\overline{\partial}} \Omega^{0,1} \xrightarrow{\overline{\partial}} \Omega^{0,2} \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} \Omega^{0,n} \to 0.$$

and more generally,

$$0 \to \text{holomorphic } (p,0)\text{-forms} \to \Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \xrightarrow{\overline{\partial}} \Omega^{p,2} \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} \Omega^{p,n} \to 0.$$

When is this exact?

[1] I got this wrong on the board; I fixed it here. DES