NOTES FOR 1-27

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We are headed toward an analogue of the Poincaré Lemma for the $\overline{\partial}$ operator. Our eventual goal is to prove that the sequence of sheaves

$$0 \to Hol \to C^\infty \xrightarrow{\overline{\partial}} \Omega^{0,1} \xrightarrow{\overline{\partial}} \Omega^{0,2} \to \cdots \to \Omega^{0,n} \to 0$$

is exact, where Hol is the sheaf of holomorphic functions, C^∞ is the sheaf of smooth ($\mathbb{C}$-valued) functions, and n is the dimension of the underlying complex manifold.

DIMENSION 1

Let us first examine this sequence in the 1-dimensional case. When the underlying manifold is $\mathbb{C}$, the claim is that

$$0 \to Hol \to C^\infty \xrightarrow{\overline{\partial}} \Omega^{0,1} \to 0$$

is exact. Clearly, $Hol \to C^\infty$ is injective, and we observed last class that Hol is precisely the kernel of the map $\overline{\partial} : C^\infty \to \Omega^{0,1}$, so the only difficulty in verifying exactness is in showing that $\overline{\partial} : C^\infty \to \Omega^{0,1}$ is surjective. That is, given a smooth function $g : \mathbb{C} \to \mathbb{C}$, we would like to produce a smooth function $f : \mathbb{C} \to \mathbb{C}$ with $\overline{\partial} f = g \, d\overline{z}$, or in other words,

$$\frac{\partial}{\partial \overline{z}} f = g.$$

Our first result in this direction will be the following:

Lemma. Let $g : \mathbb{C} \to \mathbb{C}$ be smooth and compactly supported. Then there exists a smooth function $f : \mathbb{C} \to \mathbb{C}$ such that $\partial f = g \, d\overline{z}$.

Before we prove this, let's look non-rigorously at the "formal nonsense" that motivates it. Let $g = \delta_0$ be the Dirac delta function; intuitively, $g(z) = 0$ for all $z \neq 0$, but g is "so infinite" at $z = 0$ that for any disk $D \subset \mathbb{C}$, we have

$$\int_D g \, dArea = \begin{cases} 1 & \text{if } 0 \in D \\ 0 & \text{if } 0 \notin D. \end{cases}$$

Since

$$d\overline{z} \wedge dz = (dx - idy) \wedge (dx + idy) = 2i(dx \wedge dy) = (2i)dArea,$$

we can rewrite this as

$$\int_D g d\overline{z} \wedge dz = \begin{cases} 2i & \text{if } 0 \in D \\ 0 & \text{if } 0 \notin D. \end{cases}$$

Suppose that we have found a smooth function f as in the statement of the lemma. Then

$$d(f dz) = df \wedge dz = \left(\frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \overline{z}} d\overline{z} \right) \wedge dz = \frac{\partial f}{\partial \overline{z}} d\overline{z} \wedge dz = g d\overline{z} \wedge dz.$$

So, applying Stokes's Theorem, one finds that for any disk $D \subset \mathbb{C}$,

$$\int_{\partial D} f dz = \int_D d(f dz) = \int_D g d\overline{z} \wedge dz = \begin{cases} 2i & \text{if } 0 \in D \\ 0 & \text{if } 0 \notin D. \end{cases}$$

On the other hand, the function $f(z) = 1/(\pi z)$ has precisely these same integrals around disks. So (still speaking non-rigorously), we can say that

$$\frac{1}{\pi} \overline{\partial} \left(\frac{1}{z} \right) = \delta_0 d\overline{z},$$

or more generally,

$$\frac{1}{\pi}\overline{\partial}\left(\frac{1}{z-\zeta}\right) = \delta_\zeta d\overline{z},$$

for any complex number ζ.

In an attempt to generalize this method to smooth functions other than Dirac delta functions, observe that

$$g(z) = \int_{\zeta\in\mathbb{C}} g(\zeta)\delta_\zeta(z) \, dArea$$

for any smooth function g. The heuristic discussion above would thus suggest that the function

$$f(z) = \int_{\zeta\in\mathbb{C}} g(\zeta)\frac{1}{\pi(z-\zeta)} \, dArea$$

satisfies $\overline{\partial}f = g\,d\overline{z}$, proving the lemma. Of course, before we can check that f has this property, we will need to verify that the integral in its definition is convergent, and that the resulting function is smooth. Let us now do this rigorously.

Proof of Lemma. First, re-write the potential antiderivative f as

$$f(z) = \frac{1}{2\pi i}\int_{\zeta\in\mathbb{C}} \frac{g(\zeta)}{z-\zeta}d\overline{\zeta}\wedge d\zeta.$$

Now, apply the change of variables $\eta = z - \zeta$, which moves the pole of the integrand to the origin:

$$f(z) = \frac{1}{2\pi i}\int_{\eta\in\mathbb{C}} \frac{g(z-\eta)}{\eta}d\overline{\eta}\wedge d\eta.$$

We must check that this integral converges. Since g is compactly supported, there is no convergence issue near ∞, but only near the pole of the integrand at the origin. Thus, it suffices to integrate over a small disk D_ε of radius ε centered at the origin and verify that the resulting integral converges. The fact that g is compactly supported implies that it is bounded, so $g(z-\eta) = O(1)$, whereas $\frac{1}{\eta} = O\left(\frac{1}{z}\right)$. So, switching to polar coordinates, we find that

$$\int_{\eta\in D_\varepsilon} \frac{g(z-\eta)}{\eta}d\overline{\eta}\wedge d\eta = \int_0^\varepsilon \int_0^{2\pi} O(1/r)rdrd\theta = \int_0^\varepsilon O(1)dr,$$

which is convergent.

This shows that f is a well-defined function; we must also show that it is smooth. Now, by definition, an integral whose integrand has a pole at the origin is given by

$$f(z) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{\eta\in\mathbb{C}\backslash D_\varepsilon} \frac{g(z-\eta)}{\eta}d\overline{\eta}\wedge d\eta,$$

where D_ε is a small ball of radius ε centered at the origin. The convergence of the integral is uniform, so given any differential operator $\frac{\partial}{(\partial x)^k}\frac{\partial}{(\partial y)^\ell}$, we can differentiate under the integral to give

$$\frac{\partial}{(\partial x)^k}\frac{\partial}{(\partial y)^\ell}f = \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{\mathbb{C}\backslash D_\varepsilon} \frac{\frac{\partial}{(\partial x)^k}\frac{\partial}{(\partial y)^k}g(z-\eta)}{\eta}d\overline{\eta}\wedge d\eta,$$

and the same argument as above shows that this integral converges, hence the derivative of f exists.

Finally, we will check that $\frac{\partial}{\partial\overline{z}}f = g$. Again differentiating under the integral, we have

$$\frac{\partial}{\partial\overline{z}}f = \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\mathbb{C}\backslash D_\varepsilon} \frac{\partial g(z-\eta)/\partial\overline{z}}{\eta}d\overline{\eta}\wedge d\eta.$$

Observe that

$$
\begin{aligned}
d\left(\frac{g(z-\eta)}{\eta}d\eta\right) &= \partial\left(\frac{g(z-\eta)}{\eta}d\eta\right) + \overline{\partial}\left(\frac{g(z-\eta)}{\eta}d\eta\right) \\
&= \frac{\partial}{\partial\eta}\left(\frac{g(z-\eta)}{\eta}\right)d\eta \wedge d\eta + \frac{\partial}{\partial\overline{\eta}}\left(\frac{g(z-\eta)}{\eta}\right)d\overline{\eta} \wedge d\eta \\
&= 0 + \frac{\partial g(z-\eta)/\partial\overline{\eta}}{\eta}d\overline{\eta} \wedge d\eta \\
&= -\frac{\partial g(z-\eta)/\partial\overline{z}}{\eta} \wedge d\eta.
\end{aligned}
$$

Thus, Stokes's Theorem gives

$$
\frac{1}{2\pi i}\int_{\mathbb{C}\setminus D_\varepsilon}\frac{\partial g(z-\eta)/\partial\overline{z}}{\eta}d\overline{\eta} \wedge d\eta = -\frac{1}{2\pi i}\int_{\mathbb{C}\setminus D_\varepsilon}d\left(\frac{g(z-\eta)}{\eta}d\eta\right) = \frac{1}{2\pi i}\int_{\partial D_\varepsilon}\frac{g(z-\eta)}{\eta}d\eta,
$$

where the reversal in sign is due to the fact that ∂D_ε, as the boundary of $\mathbb{C}\setminus D_\varepsilon$, receives the opposite of its standard orientation. Under the change of coordinates $\eta = \varepsilon e^{i\theta}$, this becomes

$$
\frac{1}{2\pi i}\int_0^{2\pi}\frac{g(z-\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}i\varepsilon e^{i\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi}g(z-\varepsilon e^{i\theta})d\theta,
$$

the "average value" of g on a circle of radius ε around z. In the limit as $\varepsilon \to 0$, this average value approaches $g(z)$ itself. Thus,

$$
\frac{\partial}{\partial\overline{z}}f = \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\mathbb{C}\setminus D_\varepsilon}\frac{\partial g(z-\eta)/\partial\overline{z}}{\eta}d\overline{\eta} \wedge d\eta = g,
$$

as required. $\square$

Eventually, we will get rid of the hypothesis that g is compactly supported. For now, we will only relax it slightly:

Lemma. Let $K \subset \mathbb{C}$, and let U be an open set containing K. Let $g : U \to \mathbb{C}$ be a smooth function. Then there exists an open set V with $U \supset V \supset K$ and a smooth function f on V such that $\frac{\partial f}{\partial\overline{z}} = g$ on V.

Proof. Use a bump function; that is, find a smooth function θ and a set V lying between U and K such that $\theta|_V = 1$ and $\theta|_{\mathbb{C}\setminus U} = 0$. Then $g \cdot \theta$ is compactly supported and defined on all of $\mathbb{C}$, so by the previous lemma, there exists a smooth function f such that $\frac{\partial f}{\partial\overline{z}} = g\theta$. In particular, $\frac{\partial f}{\partial\overline{z}} = g$ on V. $\square$

The main important case of the above lemma is when K is a point. For recall, our initial objective was to prove that $\overline{\partial} : C^\infty \to \Omega^{0,1}$ is surjective, which is to say that for any open set U and any $g\,d\overline{z} \in \Omega^{0,1}(U)$, there exists an open cover $\{U_i\}$ of U and smooth functions f_i on U_i for which $\overline{\partial}f_i = g\,d\overline{z}$ on U_i. To construct such an open cover, apply the lemma to each point $p \in U$ to obtain an open set $V_p \subset U$ containing p and a smooth function f_p on V_p such that $\overline{\partial}f_p = \frac{\partial f_p}{\partial\overline{z}}d\overline{z} = g\,d\overline{z}$ on V_p. Together, these V_p's form the required open cover of U, so $\overline{\partial}$ is indeed surjective.

Let us now turn to the multivariable case.

HIGHER DIMENSIONS

In the single variable case, the key requirement in the Poincaré Lemma (and its analogue for $\overline{\partial}$, considered above) was contractibility of the domain. When we consider higher dimensions, however, exactness of the sequence

$$
0 \to Hol \to C^\infty \xrightarrow{\overline{\partial}} \Omega^{0,1} \xrightarrow{\overline{\partial}} \Omega^{0.2} \to \cdots \to \Omega^{0,n} \to 0
$$

will require a more subtle condition on the domain, one that is not even topological in nature. The type of domain for which we will prove exactness is called a polydisk:

Definition. Given positive real numbers $(r_1, \ldots, r_n)$ (some possibly equal to ∞), the **polydisk** with radii $(r_1, \ldots, r_n)$ is

$$B_{r_1, \ldots, r_n} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i| < r_i\}.$$

Lemma. Let U be an open polydisk and let K be a compact polydisk in U. Let ω be a (p, q)-form on U with $\overline{\partial}\omega = 0$. Then there exists an open polydisk V with $K \subset V \subset U$ and a $(p, q-1)$-form θ on V such that $\overline{\partial}\theta = \omega$ on V.

Proof. First, we will reduce to the case $p = 0$. To do this, let ω be a (p, q)-form on U with $\overline{\partial}\omega = 0$, and write

$$\omega = \sum_I \omega_I \wedge dz_{i_1} \wedge \ldots \wedge dz_{i_p},$$

where each ω_I is a $(0, q)$-form. Notice that

$$\overline{\partial}\omega = \sum_I (\overline{\partial}\omega_I) dz_{i_1} \wedge \ldots \wedge dz_{i_p},$$

so $\overline{\partial}\omega = 0$ if and only if $\overline{\partial}\omega_I = 0$ for all I. Assuming we have proved the $p = 0$ case, we can find θ_I on smaller polydisks V_I such that $\overline{\partial}\theta_I = \omega_I$. Shrinking to a polydisk V inside all of the V_I's, define

$$\theta = \sum_I \theta_I dz_{i_1} \wedge \ldots \wedge dz_{i_p}$$

on V. Then $\overline{\partial}\theta = \omega$ on V, as required.

The proof of the $p = 0$ case is by induction on the largest index k such that $d\overline{z_k}$ appears in ω. In the base case, no $d\overline{z_k}$ appears in ω and hence $\omega = 0$, so we can take $\theta = 0$. For the inductive step, write

$$\omega = \sum_{1 \leq i_1, \ldots, i_q \leq k} f_I d\overline{z_{i_1}} \wedge \ldots \wedge d\overline{z_{i_q}}.$$

For any $\ell > k$, the coefficient of $d\overline{z_{i_1}} \wedge \ldots \wedge d\overline{z_{i_q}} \wedge d\overline{z_\ell}$ in $\overline{\partial}\omega$ is $\frac{\partial f_I}{\partial \overline{z_\ell}}$. Thus, the assumption that $\overline{\partial}\omega = 0$ implies that for all multi-indices I and all $\ell > k$, we have

$$\frac{\partial f_I}{\partial \overline{z_\ell}} = 0.$$

That is, each f_I is holomorphic in the variables $z_{k+1}, \ldots, z_n$.

By shrinking U if necessary, we may multiply by a bump function in one variable to ensure that ω extends to a smooth $(0, q)$-form on

$$U_1 \times U_2 \times \ldots \times U_{k-1} \times \mathbb{C} \times U_{k+1} \times \ldots \times U_n$$

which is supported on

$$U_1 \times U_2 \times \ldots \times U_{k-1} \times K \times U_{k+1} \times \ldots \times U_n$$

with K compact. Define a $(0, q-1)$-form α by

$$\alpha = \sum_{I \mid k \in I} \left(\frac{1}{2\pi i} \int_{\eta \in \mathbb{C}} \frac{f_I(z_1, \ldots, z_{k-1}, z_k - \eta, z_{k+1}, \ldots, z_n)}{\eta} d\overline{\eta} \wedge d\eta \right) d\overline{z_{i_1}} \wedge \ldots \widehat{d\overline{z_{i_k}}} \wedge \ldots \wedge d\overline{z_{i_q}}.$$

Notice that the $\frac{\partial}{\partial \overline{z_k}}$ terms in $\overline{\partial}\alpha$ exactly equal the $f_I d\overline{z_{i_1}} \wedge \ldots \wedge \overline{z_{i_q}}$ terms in ω with $k \in I$. On the other hand, the $\frac{\partial}{\partial \overline{z_\ell}}$ terms in $\overline{\partial}\alpha$ for $\ell > k$ are all zero, as can be seen by differentiating inside the integral and using the fact that $\frac{\partial f_I}{\partial \overline{z_\ell}} = 0$ for all I and all $\ell > k$. Thus,

$$\omega - \overline{\partial}\alpha$$

contains only $d\overline{z_i}$ terms for $i < k$. By induction, there exists a polydisk V between U and K and a $(0, q-1)$-form β on V such that $\omega - \overline{\partial}\alpha = \overline{\partial}\beta$. Hence

$$\omega = \overline{\partial}(\alpha + \beta)$$

on V, and we are done. $\qquad\square$

As in the single-variable case, taking K to be a point in the above lemma proves that the Dolbeault complex

$$\Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{p,q} \to 0$$

is exact on $\mathbb{C}^n$. In fact, since the entire argument was local, it follows that the Dolbeault complex is exact on any complex manifold.

When $p = 0$, the kernel of $\overline{\partial} : \Omega^{p,0} \to \Omega^{p,1}$ consists precisely of the holomorphic functions. In general, elements of the kernel of $\overline{\partial} : \Omega^{p,0} \to \Omega^{p,1}$ are called holomorphic $(p,0)$-forms.

Definition. A $(p,0)$-form $\sum f_I dz_{i_1} \wedge dz_{i_p}$ is called ***holomorphic*** if each f_I is a holomorphic function.

The holmorphic p-forms are precisely the kernel of $\Omega^{p,0} \to \Omega^{p,1}$.

Preview of What's To Come

Notice that $\Omega^{p,q}$ is a C^∞ module, and C^∞ has partitions of unity. Thus, for any complex manifold X, we have

$$H^k(X, \Omega^{p,q}) = 0$$

for all $k > 0$. So the Dolbeault complex

$$\Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{p,q} \to 0$$

is a resolution of $\Omega^{p,0}$ by acyclic sheaves. This implies that, if $\mathcal{H}^p$ is the sheaf of holomorphic $(p,0)$-forms on X, then

$$H^q(X, \mathcal{H}^p) = \frac{\ker(\overline{\partial} : \Omega^{p,q} \to \Omega^{p,q+1})}{\operatorname{im}(\overline{\partial} : \Omega^{p,q-1} \to \Omega^{p,q})}.$$

We would like to apply a similar trick to the complex

$$0 \to LC_\mathbb{C} \to \mathcal{H}^0 \xrightarrow{\partial} \mathcal{H}^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{H}^n \to 0,$$

but we will first need to know that this complex is exact. The proof is essentially the same as previously, with ∂ in place of $\overline{\partial}$. I'll say more about this next time.

Another goal for the coming lectures is to know some cases in which $H^k(X, \mathcal{H}^p) = 0$, or at least to know what it is when it is nonzero. For example, we will show that

$$H^k(\mathbb{C}^n, \mathcal{H}^0) = 0.$$

This amounts to showing that the complex

$$0 \to \mathcal{H}^0(\mathbb{C}^n) \to C^\infty(\mathbb{C}^n) \xrightarrow{\overline{\partial}} \Omega^{0,1}(\mathbb{C}^n) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{0,n}(\mathbb{C}^n) \to 0$$

is exact. Notice that this does *not* follow from what we have already done, since we allowed ourselves to shrink the domain when constructing a $\overline{\partial}$-antiderivative of a (p,q)-form. In order to construct global $\overline{\partial}$-antiderivatives, we will need to get rid of the compactly-supported hypothesis in all of today's results.

To summarize the concern, suppose that we are given a smooth function g on $\mathbb{C}$ and we want to find a smooth function f such that $\partial f / \partial \overline{z} = g$. We can do this when g is compactly-supported, so restricting g to concentric disks about the origin gives a function f on each disk with $\partial f / \partial \overline{z} = g$. But there is no guarantee that these functions are compatible; in fact, we might have made a particularly stupid choice of antiderivative f on some disk, such that f cannot be extended to an antiderivative on a larger disk. For example, if $g = 0$, we are trying to show that there is an entire holomorphic function. Of course, there are plenty of them, but there are also holomorphic function on discs which don't extend to global holomorphic functions. We will need to be more careful when choosing antiderivatives if they are to stand a chance of fitting together into a global antiderivative.