NOTES FOR MARCH 10

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This material coincides with chapter 5 in Voisin's book. Last time we looked at X a smooth manifold with positive definite symmetric bilinear form on T_*X and defined

$$* : \Omega^k \to \Omega^{n-k} \text{ (a map of vector bundles)}$$
$$d^* : \Omega^k \to \Omega^{k-1}$$
$$\Delta_d = dd^* + d^*d.$$

If X was compact, we also defined a positive definite symmetric form $(,) : \Omega^k \times \Omega^k \to \mathbb{R}$ such that d and d^* are adjoint. We also stated the Hodge theorem:

Theorem (Hodge)**.**

$$\ker(\Delta_d : \Omega^k \to \Omega^k) \cong H^k_{DR}(X)$$

Now let X be a complex n-fold (real dimension $2n$), with a positive definite symmetric form on T_*X such that $g(u, v) = g(Ju, Jv)$. (By partitions of unity, such a g always exists). We can then define $*, d^*$, and Δ_d as before. E.g. take $X = \mathbb{C}$ with coordinates x and y and the standard inner product. Then $* : dx \mapsto dy$ and $dy \mapsto -dx$. Extend $*$ and d^* to be $\mathbb{C}$-linear (that is to say, they act on real and imaginary parts separately) we have $*(dx + idy) = dy - idx = (-i)(dx + idy)$. In other words, $* : dz \mapsto (-i)dz$. In general we have $* : \Omega^{p,q} \to \Omega^{n-q,n-p}$.

Recall that $d = \partial + \bar{\partial}$, $d^* = - * d *$ (even dimension). We also define $\partial^* = - * \bar{\partial} *$ and $\bar{\partial}^* = - * \partial *$, so that $\partial^*\partial : \Omega^{p,q} \to \Omega^{p,q}$ and similarly for $\partial\partial^*, \bar{\partial}^*\bar{\partial}, \bar{\partial}\bar{\partial}^*$ (see Figure 1).

Now say X is compact. We'll put a positive definite Hermitian form on $\Omega^{p,q}(X)$. This is defined by

$$(\alpha, \beta) = \int_X \alpha \wedge \overline{*\beta}.$$

I can use this formula to put a Hermitian inner product on all of $\Omega^k(X)$, but it just decomposes into a sum of these as the inner product between forms of different type is 0. If α, β are real k-forms then this is just the old $(,)$. We know that $\partial + \bar{\partial}$ is adjoint to $\partial^* + \bar{\partial}^*$. In fact $(\partial\alpha, \beta) = (\alpha, \partial^*\beta)$, and $(\bar{\partial}\alpha, \beta) = (\alpha, \bar{\partial}^*\beta)$. We can also define

$$\Delta_\partial = \partial\partial^* + \partial^*\partial \text{ and } \Delta_{\bar{\partial}}\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

Note $\Delta_d = (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial}) = \Delta_\partial + \Delta_{\bar{\partial}} + \bar{\partial}\partial^* + \cdots$.

We now generalize the above ideas to holomorphic vector bundles. (So the earlier material is the case where E is trivial.)

Let $E \to X$ be a $\mathbb{C}$ vector bundle with a positive definite Hermitian form. We have one pairing

$$(E \otimes \Omega^{p,q}) \times (E \otimes \Omega^{p,q}) \to \underline{\mathbb{C}}$$

which is sesquilinear. This is $\langle \sigma \otimes \omega, \tau \otimes \eta \rangle = \langle \sigma, \tau \rangle \langle \omega, \eta \rangle$. We also have

$$(E \otimes \Omega^{p,q}) \times (E^* \otimes \Omega^{n-p,n-q}) \to \Omega^{n,n} \cong \underline{\mathbb{C}}$$

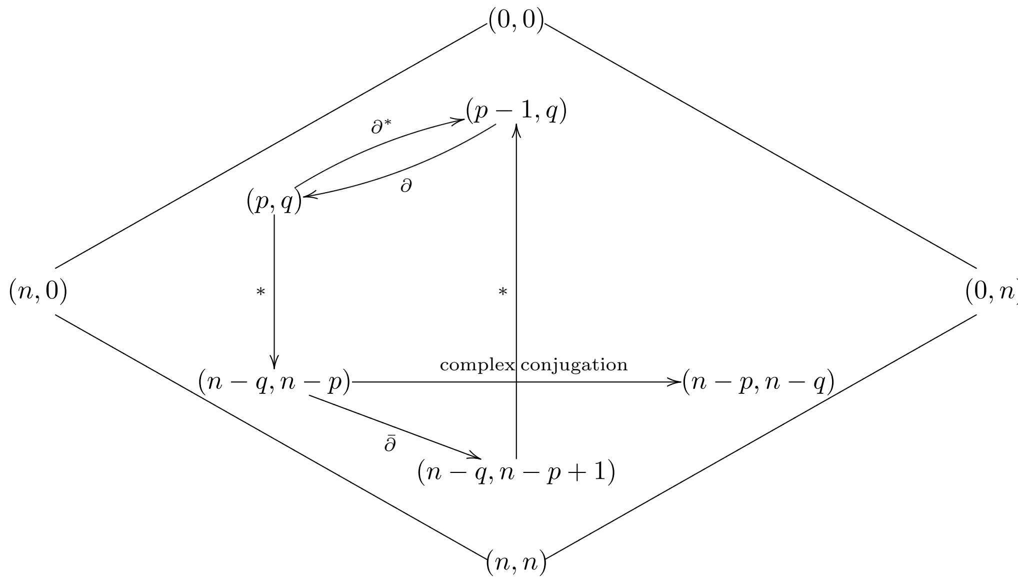

FIGURE 1. Our various maps in the Hodge diamond

We get the trivialization of $\Omega^{n,n}$ by using the inner product to get a volume form up to sign and using the complex structure to get an orientation.

As in the previous lecture, given two perfect pairings, there is a linear map making them coincide. This map is $*_E : E \otimes \Omega^{p,q} \to E^* \otimes \Omega^{n-p,n-q}$ and it is $\mathbb{C}$-antilinear. If $E = \underline{\mathbb{C}}$ with the standard Hermitian form then $*_E = \bar{*}$.

Remark: On the trivial vector bundle, we have $*$, which is $\mathbb{C}$-linear, and we have complex conjugation. (A k-form eats k vectors and spits out a complex number; its complex conjugate eats the same k vectors and spits out the complex conjugate number.) On a general vector bundle, we don't have a notion of complex conjugate, nor of $*$, only their composition $*_E$.

If E is holomorphic then it comes with a natural $\bar{\partial}$-connection. $\bar{\partial} : E \otimes \Omega^{p,q} \to E \otimes \Omega^{p,q+1}$. We also define $\bar{\partial}^*_E = (-1)^q *^{-1} \bar{\partial}_{E^* \otimes K} *$ where K is $\Omega^{n,0}$ considered as a holomorphic vector bundle. In other words, $\bar{\partial}^*_E$ is

$$E \otimes \Omega^{0,q} \overset{*_E}{\to} E^* \otimes \Omega^{n,n-q} = E^* \otimes \Omega^{n,0} \otimes \Omega^{0,n-q} = (E^* \otimes K) \otimes \Omega^{0,n-q}$$

$$\overset{\bar{\partial}_{E^* \otimes K}}{\to} (E^* \otimes K) \otimes \Omega^{0,n-q+1} \to \cdots \overset{*^{-1}}{\to} E \otimes \Omega^{0,q-1}$$

where K is the canonical bundle. Similarly, we have maps $\bar{\partial} : E \otimes \Omega^{p,q} \to E \otimes \Omega^{p,q-1}$.

If X is compact, we get a Hermitian positive definite inner product on $E \otimes \Omega^{p,q}(X)$ with $\bar{\partial}_E$ and $\bar{\partial}^*_E$ adjoint. As before, $\Delta_E = \bar{\partial}_E \bar{\partial}^*_E + \bar{\partial}^*_E \bar{\partial}_E$.

Theorem.

$$\ker(\Delta_E : (E \otimes \Omega^{p,q})(X) \to (E \otimes \Omega^{p,q})(X))$$

$$\cong \frac{\bar{\partial} - \textit{closed } (E \otimes \Omega^{p,q})(X)}{\bar{\partial} - \textit{exact } (E \otimes \Omega^{p,q})(X)} \cong H^q(X, \mathrm{Hol}(E) \otimes \mathcal{H}^p).$$

Furthermore, these vector spaces are finite dimensional.

The proof is analogous to the proof of the Hodge theorem. Roughly speaking, one studies the eigenspaces of Δ_E, but the analytic details are again very subtle. There is no simple proof known of the finite dimensionality of $H^q(X, \mathrm{Hol}(E))$.

Poincaré and Serre Duality

We return to the world of smooth manifolds. Let X be a compact, smooth, oriented n-fold. Define a pairing $\Omega^k(X) \times \Omega^{n-k}(X) \to \mathbb{R}$ by $(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$. If α is closed and we change β to $\beta + d\gamma$ then

$$\int \alpha \wedge (\beta + d\gamma) = \int \alpha \wedge \beta + \int \alpha \wedge d\gamma = \int \alpha \wedge \beta + \int d(\alpha \wedge \gamma) = \int \alpha \wedge \beta$$

so we get a pairing $H^k_{DR}(X) \times H^k_{DR}(X) \to \mathbb{R}$.

Theorem (Poincaré duality)**.** *This paring is perfect.*

Proof. Given $\alpha \in H^k_{DR}$ nonzero, we want $\beta \in H^{n-k}_{DR}$ such that $\int \alpha \wedge \beta \neq 0$. Choose a metric on X. Lift α to $\tilde{\alpha}$ in $\ker \Delta$. Then $d\tilde{\alpha} = 0$ and $d^*\tilde{\alpha} = 0$. So $d(*\tilde{\alpha}) = \pm * d^*\tilde{\alpha} = 0$ so $*\tilde{\alpha}$ is d-closed. And $\int \tilde{\alpha} \wedge (*\tilde{\alpha}) > 0$ (the inequality is strict because $\alpha \neq 0$ means $\tilde{\alpha} \neq 0$). ☐

An extremely similar proof does Serre duality. Let E be a holomorphic vector bundle over complex compact n-fold X.

Theorem (Serre duality)**.** *$H^q(X, \mathrm{Hol}(E))$ and $H^{n-q}(X, K \otimes \mathrm{Hol}(E))$ are dual.*

The way the pairing goes is, given α a $\bar{\partial}$-closed $(0, q)$-form valued in E, and β a $\bar{\partial}$-closed $(0, n-q)$-form valued in $E^* \otimes K$, we wedge them and pair together $E \otimes E^*$ to get $\alpha \wedge \beta$, a $(0, n)$ form valued in K or, in otherwords, an (n, n)-form. Our pairing then sends (α, β) to $\int_X \alpha \wedge \beta$. Note that this pairing is $\mathbb{C}$-linear.

The analogous theorem in the algebraic setting is that $H^n(X, K) \cong \mathbb{C}$ canonically and

$$H^q(X, E) \times H^{n-q}(X, E^* \otimes K) \to H^n(X, K) \cong \mathbb{C}$$

is a $\mathbb{C}$-linear perfect pairing. The isomorphism $H^n(X, K) \cong \mathbb{C}$ is rather subtle when you have no analytic tools.

Proof. Given $\alpha \in H^q(X, \mathrm{Hol}(E))$ we can lift to $\tilde{\alpha} \in (E \otimes \Omega^{0,q})(X)$ which is in $\ker \Delta_E$. As before,

$$(\tilde{\alpha}, \Delta_E \tilde{\alpha}) = (\bar{\partial}_E \tilde{\alpha}, \bar{\partial}_E \tilde{\alpha}) + (\bar{\partial}^*_E \tilde{\alpha}, \bar{\partial}^*_E \tilde{\alpha})$$

so $\bar{\partial}_E \tilde{\alpha} = \bar{\partial}^*_E \tilde{\alpha} = 0$. Let $\beta = *_E \tilde{\alpha} \in (E^* \otimes K \otimes \Omega^{0,n-q})(X)$. Then $\bar{\partial}_K \otimes E^* \beta = \pm * \bar{\partial}^*_E \tilde{\alpha} = 0$. So $*\tilde{\alpha}$ gives a class in $H^{n-q}(X, K \otimes E^*)$ and $\int \tilde{\alpha} \wedge *\tilde{\alpha} > 0$. ☐

Remark: We used a $\mathbb{C}$-antilinear map, namely $*_E$, to prove a $\mathbb{C}$-linear duality.

A very brief example of the Hodge theorem

On $\mathbb{R}^2$ with the ordinary metric, everything real, we have

$$f \overset{d}{\mapsto} \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \overset{*}{\mapsto} -\frac{\partial f}{\partial y} dx + \frac{\partial f}{\partial x} dy$$

$$\overset{d}{\mapsto} \frac{\partial^2 f}{\partial y^2} dx \wedge dy + \frac{\partial^2 f}{\partial x^2} dx \wedge dy \overset{*^{-1}}{\mapsto} -\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right).$$

Let's look at the torus $T = \mathbb{R}^2/\mathbb{Z}^2$, with the quotient metric. The eigenfunctions of Δ look like $\cos(2\pi ax)\cos(2\pi bx)$, and likewise with one or both cosines replaced by a sine. So the eigenvalues of Δ_d are $(4\pi^2)(a^2 + b^2)$.

We see that this is, indeed, a discrete sequence of nonnegative reals. In particular, $\mathcal{K}er\Delta$ is 1-dimensional, spanned by 1 (corresponding to $(a, b) = (0, 0)$). This is because T is connected.

Similarly, the eigenforms of Δ on 1-forms look like $\cos(2\pi ax)\cos(2\pi bx)dx$ and $\cos(2\pi ax)\cos(2\pi bx)dy$. The kernel of Δ is spanned by dx and dy, verifying that $H^1(T)$ is 2-dimensional. On 2-forms, the kernel of Δ is spanned by $dx \wedge dy$.

This is the only easy case of the Hodge theorem. The sphere S^n and projective space $\mathbb{CP}^n$ can be done by hand if you work hard enough. In basically every other case, the eigenfunctions/eigenforms are beyond the limits of reasonable computation.