MATH 632 NOTES: KÄHLER MANIFOLDS

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Definition 0.1. Let X be a complex manifold equipped with a J-invariant positive definite Hermitian form $\langle\ ,\ \rangle = g - i\omega$. We say X is *Kähler* if $d\omega = 0$.

We might also say $\langle\ ,\ \rangle$, g, or ω are Kähler. For example, ω is Kähler if

- $\omega(Ju, Jv) = \omega(u, v)$,
- $\omega(u, JV)$ is positive definite,
- $d\omega = 0$.

Note: Kähler is a local condition.

Differential (Riemannian) geometry perspective: start with (X, g). Then we get a connection ∇_{LC} on T_*X called "Levi-Cevita." This is the unique connection such that

(1) ∇ preserves the metric,
(2) ∇ has no torsion: if X and Y are commuting vector fields, then $\nabla_X Y = \nabla_Y X$.

Now the Kähler condition means that the Levi-Cevita connection commutes with J: $J\nabla_{LC} v = \nabla_{LC} Jv$.

Observe that if X is a Kähler complex manifold and Y is a complex submanifold, then Y is Kähler in the restricted Hermitian form. This is just becaue d commutes with restriction.

Example 0.1. Let X be any curve over $\mathbb{C}$ and $\langle\ ,\ \rangle$ any Hermitian form on T_*X. Then $d\omega$ is a 3-form, so $\dim_{\mathbb{R}} X = 2$ forces $d\omega = 0$.

Example 0.2. Let V be an n-dimensional complex vector space and identify $T_x V = V$. If $\Lambda \cong \mathbb{Z}^{2n}$ is a discrete lattice in V, then any form descends to V/Λ, the complex torus.

Example 0.3. Let V be an n-dimensional complex vector space with a positive definite Hermitian form. Recall that $\mathbb{P}(V) = (V \setminus \{0\})/\mathbb{C}^\times$. Define $S(V) = \{v \in V \mid \langle v, v\rangle = 1\} \cong S^{2n-1}$, so that $\mathbb{P}(V) = S(V)/S^1$. Given $x \in \mathbb{P}(V)$, we want a Hermitian form on $T_x\mathbb{P}(V)$. Lift x to $\widetilde{x} \in S(V)$. We have a short exact sequence

$$0 \to T_{\widetilde{x}}(S^1\widetilde{x}) \to T_{\widetilde{x}}S(V) \to T_x\mathbb{P}(V) \to 0,$$

and we can use the inner product to identify $T_x\mathbb{P}(V) \cong T_{\widetilde{x}}(S^1\widetilde{x})^\perp$. We have

$$T_{\widetilde{x}}S(V) = \{y \in V \mid \operatorname{Re}\langle y, \widetilde{x}\rangle = 0\}$$

and $T_{\widetilde{x}}(S^1\widetilde{x}) = \mathbb{R}\cdot(i\widetilde{x})$, so

$$T_x\mathbb{P}(V) \cong y \in V \mid \langle y, \widetilde{x}\rangle = 0\}.$$

If you change the lift $\widetilde{x}$ by $e^{i\theta}$, the isomorphism changes by $e^{i\theta}$. This doesn't affect the restriction of $\langle\ ,\ \rangle$ to the right hand side, so we get the so-called Fubini-Study form on $\mathbb{P}(V)$. Now $\mathbb{P}(V)$ has an action of $\mathbb{P}\operatorname{GL}(V)$, but note that the Fubini-Study form is only invariant under $\mathbb{P}U(V)$.

Theorem 0.1. *The Fubini-Study form is Kähler.*

We'll see soon that any complex line bundle gives rise to a closed J-invariant 2-form, which will imply the theorem.

Lemma 0.2 (Nice coordinates)**.** *If X is a complex manifold with a positive definite Hermitian form $\langle\ ,\ \rangle$, then X is Kähler if and only if for every $x \in X$, there are holomorphic coordinates $(z_1, \cdots, z_n)$ near x such that $\langle\ ,\ \rangle = \sum a_{ij}(z)\ dz_i \otimes d\overline{z}_j$, where $a_{ij} = \delta_{ij} + O(\sum |z_k|^2)$ near x.*

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Of course $\langle\ ,\ \rangle$ is in the span of $dz_i \times d\overline{z}_j$ because it is Hermitian. Without error terms, the requirement is that we are looking at the standard form on $\mathbb{C}^n$ pulled back by the $(z_1, \cdots, z_n)$.

(The following point was explained badly in class) Since $\langle\ ,\ \rangle$ is a smooth inner product, the a_{ij} will be smooth functions. Therefore, the condition that $a_{ij} - \delta_{ij}$ vanishes to order 2 implies that its derivatives vanish to order 1. (Proof: The condition that $F(x) = O(\sum |x_k|^2)$ implies that $\partial F/\partial x_i = 0$ at 0. Since F is smooth, so is $\partial F/\partial x_i$ and, thus $(\partial F/\partial x_i)(x) = (\partial F/\partial x_i)(0) + O(\sum |x_i|)$.

If we didn't have F smooth, it might be something like $x^2 \sin(1/x)$, which vanishes to order 2 but whose derivative is discontinuous at 0. It might even be something like $e^{-1/x^2}\sin(e^{2/x^2})$, which vanishes faster than any polynomial at 0, yet has discontinuous derivative. But smoothness eliminates all that.

Proof. '$\Leftarrow$' We care about

$$\omega = \sum a_{ij}\ dz_i \wedge d\overline{z}_j,$$

so write

$$d\omega = \sum da_{ij}\ dz_i \wedge d\overline{z}_j = \sum O(\sum |z_k|)\ dz_i \wedge d\overline{z}_j$$

at x. Thus $d\omega = 0$ at any $x \in X$.

'$\Rightarrow$' In any coordinates, sesquilinearity forces

$$\langle\ ,\ \rangle = \sum b_{ij}\ d\omega_i \otimes d\overline{\omega}_j.$$

Find holomorphic functions y_i such that $\frac{\partial}{\partial y_j}$ are orthonormal at x. This gives

$$\langle\ ,\ \rangle = \sum dy_i \otimes d\overline{y}_j + \sum c_{ij} dy_i \otimes d\overline{y}_j$$

with the c_{ij} smooth and $c_{ij}(x) = 0$. Then

$$\omega = \frac{1}{2i}(\sum dy_i \wedge d\overline{y}_j + \sum c_{ij}\ dy_i \wedge d\overline{y}_j).$$

By hypothesis, $d\omega = 0$, so $\partial = 0$, i.e.

$$\sum \frac{\partial c_{ij}}{\partial y_k}\ dy_k \wedge dy_i \wedge d\overline{y}_j = 0.$$

This says that $\frac{\partial c_{ij}}{\partial y_k} = \frac{\partial c_{kj}}{\partial y_i}$. We can deduce that there is a quadratic function

$$\varphi_j = \frac{1}{2}\sum \left.\frac{\partial c_{ij}}{\partial y_k}\right|_x y_i y_k$$

such that $\frac{\partial \varphi_j}{\partial y_i}|_x = c_{ij}|_x$. Put $z_j = y_j + \varphi_j(y)$, and since φ_j is holomorphic these are holomorphic coordinates. Note

$$dz_j = dy_j + \sum \frac{\partial \varphi_j}{\partial y_k}\ dy_k,$$

so

$$\begin{aligned}
\sum dz_j \wedge d\overline{z}_j &= \sum (dz_j + \sum \frac{\partial \varphi_j}{\partial y_k}\ dy_k) \wedge (d\overline{y}_j + \sum \overline{\frac{\partial \varphi_j}{\partial y_k}}\ d\overline{y}_k) \\
&= \sum dy_j \wedge d\overline{y}_j + \sum \frac{\partial \varphi_j}{\partial y_k}\ dy_k \wedge d\overline{y}_j + \cdots + \sum \frac{\partial \varphi_j}{\partial y_k}\overline{\frac{\partial \varphi_j}{\partial y_\ell}}\ dy_k \wedge d\overline{y}_\ell \\
&= \sum dy_j \wedge d\overline{y}_j + \sum c_{ij}\ dy_i \wedge d\overline{y}_j + O(\sum |y_i|^2) \\
&= \omega + O(\sum |y_i|^2).
\end{aligned}$$

But $(y_1, \cdots, y_n) \to (z_1, \cdots, z_n)$ has a local smooth inverse, so $O(\sum |y_i|^2) = O(\sum |z_i|^2)$.

$\square$