

NOTES FOR MARCH 17

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1. THE KÄHLER IDENTITIES

Suppose X is a Kähler manifold with inner product $\langle, \rangle = s - i\omega$. Recall that the condition that X be Kähler is that $d\omega = 0$. We have a map $*$ sending (p, q) -forms to $(n - p, n - q)$ forms. We will define two new operators, L and Λ . The first is

$$L: \Omega^{p,q} \rightarrow \Omega^{p+1,q+1},$$

given by the rule $\eta \mapsto \omega \wedge \eta$. Recall that the J -invariance of ω mandates that ω is a 1-1 form.

Λ is the adjoint map to L , given by

$$\Lambda = *^{-1}L*: \Omega^{p+1,q+1} \rightarrow \Omega^{p,q}.$$

Remark 1. On \mathbb{C}^n with the standard $*$, $\omega = \frac{i}{2} \sum dz_j \wedge d\bar{z}_j$. In this case, L “adds $dz_j \wedge d\bar{z}_j$ ” in front of the form in all possible ways, and Λ removes $\frac{i}{2} dz_j \wedge d\bar{z}_j$ from the front of a $(p + 1, q + 1)$ -form in all possible ways.

For any vector space V we can define the commutator $[,]$ in the usual way. Given a pair of linear maps A, B on V , $[A, B] = A \circ B - B \circ A$.

The Kähler identities are stated in terms of Ω, L and $[,]$. They are:

- i) $[\Omega, \bar{\partial}] = -i\partial^*$
- ii) $[\Omega, \partial] = i\bar{\partial}$.

We are going to state all of the above in a little more generality; this will save us from needing to reprove the Kähler identities when they are needed in a more general setting later.

Namely, let E be a rank r holomorphic vector bundle on X , a complex manifold. We have seen that E has a $\bar{\partial}$ connection; call it \bar{D} . For a smooth section σ , $\bar{D}\sigma = 0$ if and only if σ is holomorphic. Furthermore $\bar{D}^2 = 0$. Indeed, if one chooses a holomorphic basis of sections of E , then in that basis, $\bar{D} = \bar{\partial}$.

Now suppose E has a positive definite Hermitian form \langle, \rangle on E .

Lemma 1. *There exists a unique connection ∇ on E such that:*

- 1) ∇ preserves \langle, \rangle meaning

$$X \langle \sigma, \tau \rangle = \langle \nabla_X \sigma, \tau \rangle + \langle \sigma, \nabla_X \tau \rangle$$

for all vector fields X and sections σ, τ .

- 2) $\nabla = D + \bar{D}$ for a ∂ -connection D .

Proof. To see uniqueness locally, choose sections z_1, \dots, z_n on X and an orthonormal basis e_1, e_2, \dots, e_r of E . Write

$$\bar{D} = \bar{\partial} + \sum_j B_j d\bar{z}_j,$$

where B_j is an $r \times r$ matrix of smooth \mathbb{C} -valued functions.

We want to write D as $\sum A_j dz_j$ with

$$\begin{aligned} D + \bar{D} &= d + \sum A_j dz_j + \sum B_j d\bar{z}_j \\ &= d + \sum (A_j + B_j) dx_j + i \sum (A_j - B_j) dy_j. \end{aligned}$$

We have written the right hand side in terms of dx_j 's and dy_j 's so that we can compare with our previous computations for what it means to respect \langle, \rangle . Namely, we computed on February

24 that ∇ preserves \langle, \rangle if and only if $A_j + B_j$ and $i(A_j - B_j)$ are anti-Hermitian. This gives two linear equations,

$$A_j + B_j = -\bar{A}_j^t - \bar{B}_j^t \text{ and } i(A_j - B_j) = -(-i)(\bar{A}_j^t - \bar{B}_j^t).$$

Solving this pair of linear equations we get a unique solution: $A_j = -\bar{B}_j^t$. **I think I got some of these signs wrong on the blackboard.**

Hence, A_j is uniquely specified locally, and also exists locally. To see global existence, choose an open cover by U_i 's, with associated D_i 's. On any intersection $U_i \cap U_j$, D_i and D_j must agree (by uniqueness on $U_i \cap U_j$), which shows that the D_j 's glue to give a well-defined global D . \square

Note: We know $\bar{D}^2 = 0$ and $D^2 = 0$ also, so that

$$(1) \quad \nabla^2 = (D + \bar{D})^2 = D\bar{D} + \bar{D}D.$$

We explained in the previous lecture that for all E , the map ∇^2 was a map of C^∞ -modules, so that it was represented by some $\Theta \in \text{End}(E) \otimes \Omega^2(X)$, for which $\nabla^2\sigma = \Theta\sigma$.

In the case that we obtain ∇ as in the lemma, then Θ will actually be in $\text{End}(E) \otimes \Omega^{1,1}$ (this follows from equation (1)).

If we also have an inner product on X , we can also talk about $D^* = (-1)^p *_E^{-1} (D_{E^* \otimes K}) *_E$ and $\bar{D}^* = (-1)^q *_E^{-1} (\bar{D}_{E^* \otimes K}) *_E$.

Reminder: As we discussed on March 10, when E is trivial, we can just use the Hodge $*$ operator and describe ∂^* as $\pm *_E \bar{\partial}^*$. If f is a section of the trivial bundle \mathbb{C} , then $*_{\mathbb{C}}f = \pm *_E f$. For a general bundle E , we can't talk about $*$ and complex conjugation individually, only about their combination $*_E$, so things are a bit messier.

With the Kähler condition, we again have $L: E \otimes \Omega^{p,q} \rightarrow E \otimes \Omega^{p+1,q+1}$ and its adjoint, Λ . In the case E is Kähler, we can state Kähler identities as above:

Theorem 1. (*Kähler Identities*)

- i) $[\Omega, \bar{D}] = iD^*$
- ii) $[\Omega, D] = i\bar{D}^*$.

Proof. We will prove i); ii) follows by complex conjugation. Since this is an equality of sections, it suffices to check these equalities at a point x . By the work we did last class, we can pass to nice coordinates about x ; call them z_1, \dots, z_n . Also let e_1, \dots, e_r be an orthonormal basis for E . Recall that "niceness" is the statement that

$$\langle, \rangle = \sum_j (\delta_{ij} + O(\sum_k |z_k|^2)) dz_i \otimes d\bar{z}_i.$$

We claim that, in these coordinates, it suffices to check the claim when $X = \mathbb{C}^n$, with the standard Kähler form. In this case, we are verifying the equality

$$*_E^{-1} L *_E \bar{D} - \bar{D} *_E^{-1} L *_E (\sigma) = -i *_E^{-1} \bar{D} *_E \sigma.$$

We'll check that the error terms work out correctly (so that verifying this equality is sufficient to prove the identity where $*$'s are replaced by $*_E$'s): Let $\sigma \in E \otimes \Omega^{p,q}$ be given. Performing $*\sigma$ is off by $O(\sum |z_k|^2)$, so that performing $D *_E \sigma$ is off by $O(\sum |z_k|)$, so that $*^{-1} D *$ is still off by $O(\sum |z_k|)$. Namely, the value of the RHS of the above equality is independent of whether we choose the given Kähler form or the standard one.

On the LHS:

- $\bar{D}\sigma$ is the same in either Kähler form
- $*\bar{D}\sigma$ is off by $O(\sum |z_k|^2)$
- $L *_E \bar{D}\sigma$ is off by $O(\sum |z_k|^2)$
- $*L *_E \bar{D}\sigma$ is off by $O(\sum |z_k|^2)$,

while,

- $*\sigma$ is off by $O(\sum |z_k|^2)$
- $L *_E \sigma$ is off by $O(\sum |z_k|^2)$
- $*L *_E \sigma$ is off by $O(\sum |z_k|^2)$
- $\bar{D} *_E L *_E \sigma$ is off by $O(\sum |z_k|)$

The first four lines of the above show that the first term on the LHS can be computed by the standard Kähler form, and the second four lines show that the second term can be.

Now we do the computation in the standard case. Everything is \mathbb{C} -linear so suffices to do this for $\tau := f dz_I \wedge d\bar{z}_J$, for $f \in C^\infty E$. First, we examine what kind of terms show up in the identity. Label the three things that we need to compute in the identity as (A), (B), (C) from left to right, so that the Kähler identity is $(A) - (B) = (C)$.

The terms in (C) will be spanned by terms of the form $dz_{I-r} \wedge d\bar{z}_J$, where $r \in I$.

The terms in (A) will be spanned by $dz_{I-r} \wedge d\bar{z}_J$ and $dz_{I-t} \wedge d\bar{z}_{J+s-t}$.

The terms in (B) will be spanned by $dz_{I-r} \wedge d\bar{z}_J$ and $dz_{I-t} \wedge d\bar{z}_{J+s-t}$.

We claim that the terms when $s \neq t$ in (A) and (B) cancel out, while the terms where $s = t$ (so that all you do is remove r from I) add up to give the terms in (C).

First, we will show the cancellation when $s \neq t$:

We first compute the coefficient of $dz_{I-t} \wedge d\bar{z}_{J+s-t}$ in (A).

$$\begin{array}{llll} \text{The coefficient of} & dz_I \wedge d\bar{z}_J & \text{in} & \tau & \text{is} & f \\ & dz_I \wedge d\bar{z}_{J+s} & & \bar{D}\tau & & \frac{\partial f}{\partial \bar{z}_s} + B_s \cdot f \\ & dz_{I-t} \wedge d\bar{z}_{J+s-t} & & \Lambda \bar{D}\tau & & \frac{\partial f}{\partial \bar{z}_s} + B_s \cdot f \end{array}$$

Here B_s is the $r \times r$ matrix of 1-forms that shows up when you compute connections locally.

On the other hand, (B) corresponds to the sequence

$$\begin{array}{llll} \text{The coefficient of} & dz_I \wedge d\bar{z}_J & \text{in} & \tau & \text{is} & f \\ & dz_{I-t} \wedge d\bar{z}_{J-t} & & \Lambda\tau & & f \\ & dz_{I-t} \wedge d\bar{z}_{J+s-t} & & \bar{D}\Lambda\tau & & \frac{\partial f}{\partial \bar{z}_s} + B_s \cdot f \end{array}$$

This exhibits the desired cancellation in the $s \neq t$ case.

In the case that $s = t$, we have that

$$\begin{array}{llll} \text{The coefficient of} & dz_{I-r} \wedge d\bar{z}_J & \text{in} & \Lambda \bar{D}\tau & \text{is} & \begin{cases} 0 & \text{if } r \in J, r \notin I \\ \pm i(\frac{\partial f}{\partial \bar{z}_r} + B_r f) & \text{if } r \in I - J \end{cases} \\ & dz_{I-r} \wedge d\bar{z}_J & & \bar{D}\Lambda\tau & & \begin{cases} 0 & \text{if } r \notin J, r \notin I \\ \mp i(\frac{\partial f}{\partial \bar{z}_r} + B_r f) & \text{if } r \in I \cap J \end{cases} \\ & dz_{I-r} \wedge d\bar{z}_J & & D^*\tau & & \begin{cases} 0 & \text{if } r \notin I \\ \pm i(\frac{\partial f}{\partial \bar{z}_r} + B_r f) & \text{else} \end{cases} \end{array}$$

so that the terms match up, as desired. □

2. SOME CONSEQUENCES

The Kähler identities give the equality

$$D^*\bar{D} + \bar{D}D^* = i((\Lambda\bar{D} - \bar{D}\Lambda)\bar{D} + \bar{D}(\Lambda\bar{D} - \bar{D}\Lambda)) = 0,$$

which in turn shows that

$$\begin{aligned} \Delta_\nabla &= (D + \bar{D})(\bar{D}^* + D^*) + (\bar{D}^* + D^*)(D + \bar{D}) \\ &= DD^* + \bar{D}\bar{D}^* + D^*D + \bar{D}^*\bar{D} \\ &= \nabla_D + \nabla_{\bar{D}}. \end{aligned}$$

For the trivial bundle, this becomes

$$\Delta_d = \Delta_\partial + \Delta_{\bar{\partial}}.$$

Again expressing the curvature as Θ , a computation using the bracket identity $[X, YZ] = [X, Y]Z + Y[X, Z]$ and the Kähler identities shows that

$$\begin{aligned} [\Lambda, \Theta] &= [\Lambda, D\bar{D} + \bar{D}D] \\ &= [\Lambda, D]\bar{D} + D[\Lambda, \bar{D}] + \bar{D}[\Lambda, D] + [\Lambda, \bar{D}]D \\ &= -i(\bar{D}^*\bar{D} + DD^* - \bar{D}\bar{D}^* + D^*D) \\ &= i(\Delta_D - \Delta_{\bar{D}}). \end{aligned}$$

Hence, if $\Theta = 0$, for example in the important case that $E = \mathbb{C}$, then the corollary says that

$$\Delta_d = \Delta_{\bar{\partial}} + \Delta_{\partial} = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}.$$

Now consider a k -form $\alpha = \sum \alpha^{p,q}$. We have that

$$\Delta_{\partial\alpha} = 2 \sum \Delta_{\bar{\partial}}\alpha^{p,q},$$

and since $\Delta_{\bar{\partial}}$ fixes each $\Omega^{p,q}$, we have the statement that a given form is ∂ -harmonic if and only if each of its (p, q) -components is $\bar{\partial}$ -harmonic.

When X is compact, combining this fact with the the ‘‘Hodge Isomorphism’’ gives a sequence of isomorphism

$$\begin{aligned} H_{\text{DR}}^k(X; \mathbb{C}) &\cong \text{Ker}(\Delta_d: \Omega^k \rightarrow \Omega^k) \\ &= \oplus \text{Ker}(\Delta_{\bar{\partial}}: \Omega^{p,q} \rightarrow \Omega^{p,q}) \\ &\cong H^q(X, \mathcal{H}^p). \end{aligned}$$

The third term in this isomorphism is called the **Hodge Decomposition**.