NOTES FOR MARCH 22

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1. The Hodge Decomposition

Recall that for X a (possibly non-compact) Kähler manifold of (complex) dimension n, we have a bunch of operators:

$$L,\ \Lambda,\ \partial,\ \overline{\partial},\ \partial^*,\ \overline{\partial}^*,\ *,\ \Delta_d,\ \Delta_\partial,\ \text{and}\ \Delta_{\overline{\partial}}.$$

The first four and the $*$ are the most important, in that the others are built out of them. We are going to try to shove $*$ under the rug and express all operators in terms of the first four operators.

Last time, we proved Kähler identities:

$$[\Lambda, \overline{\partial}] = -i\partial^* \text{ and } [\Lambda, \partial] = i\overline{\partial}^*.$$

These let us write ∂^* and $\overline{\partial}^*$ in terms of L, Λ, ∂ and $\overline{\partial}$.

As a consequence, we saw that

$$\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d.$$

Recall that a form $\eta \in \Omega^k$ is called "harmonic" if $\Delta_d \eta = 0$; as X is Kähler, this is equivalent to saying that $\Delta_\partial \eta = 0$ or to saying that $\Delta_{\overline{\partial}}\eta = 0$. Let $\alpha \in \Omega^k$ be $\sum_{p+q=k} \alpha^{pq}$ where $\alpha^{pq} \in \Omega^{p,q}(X)$. As

$$\Delta_\partial = \partial\partial^* + \partial^*\partial,$$

it preserves the (p,q) degree, and so $\Delta_d \alpha = 0$ if and only if $\Delta_\partial \alpha = \sum \Delta_\partial \alpha^{pq} = 0$, which occurs if and only if each summand $\Delta_\partial \alpha^{pq} = 0$. Thus, a k-form α on a Kähler manifold is harmonic if and only if each of its (p,q)-components is harmonic. We'll see in the problem sets that this conclusion does *not* always hold without the Kähler hypothesis.

So far we have made only local claims about Kähler manifolds, but with a compactness condition, we get a global result. If X is compact, then we get the ***Hodge decomposition***:

$$\begin{aligned} H^k_{\mathrm{dR}}(X, \mathbb{C}) &\cong \{\text{Harmonic } k\text{-forms}\} && \text{(by the Hodge theorem)} \\ &\cong \bigoplus_{p+q=k} \{\text{Harmonic } (p,q)\text{-forms}\} && \text{(by the Kähler identities, as explained above)} \\ &\cong \bigoplus_{p+q=k} H^q(X, \mathcal{H}^p) && \text{(by the Hodge theorem for holomorphic vector bundles)} \end{aligned}$$

The first two steps say that every cohomology class in $H^k_{\mathrm{dR}}(X, \mathbb{C})$ is uniquely represented by a harmonic k-form, which is uniquely a sum of harmonic (p,q)-forms. Note that the decomposition is a claim about the complex structure on X, and not about the structure of X as a Kähler manifold! Later today (and Thursday, as it turned out), we'll prove that the decomposition is independent of the choice of Kähler structure.

***Remark* 1.1.** — Where are we using compactness in the Hodge decomposition? In step one (and step three). It's not true for non-compact X that there is an isomorphism

$$\ker \Delta \to H^k_{\mathrm{dR}}(X, \mathbb{C}).$$

We used the integration pairing on $\Omega^k(X)$ to prove the injectivity of this map (on the homework), and that pairing only makes sense if X is compact. In fact, we needed that pairing even to define the map; that is, to get that $\ker \Delta \subset \ker d$ we used that $(\Delta\eta, \eta) = (d\eta, d\eta) + (d^*\eta, d^*\eta)$.

For an example where the claim fails if X is non-compact, consider $\mathbb{R}$, which has one-dimensional cohomology in degree 0, but a 2-dimensional family of harmonic functions given by $ax+b$. Similarly, there are non-trivial harmonic 1-forms on $\mathbb{R}$ given by $(ax+b)dx$, even though the first cohomology vanishes.

2. Consequences for the cohomology of compact Kähler manifolds.

Note that complex conjugation takes harmonic (p,q)-forms to harmonic (q,p)-forms:

$$\Delta_d \overline{\eta} = \overline{\Delta_d \eta}$$
$$\Delta_\partial \overline{\eta} = \overline{\Delta_{\overline{\partial}} \eta}$$

Thus one has

$$\dim H^q(X, \mathcal{H}^p) = \dim H^p(X, \mathcal{H}^q).$$

From this, we get a four-fold symmetry in the Hodge diamond. Namely, writing $H^{p,q} := H^q(X, \mathcal{H}^p)$, Serre duality tells us that $H^{p,q}$ matches up with $H^{n-p,n-q}$, and the above observation tells us that $H^{p,q}$ matches up with $H^{q,p}$. So the dimension of the (p,q), (q,p), $(n-q, n-p)$, $(n-p, n-q)$ spots in the Hodge diamond are all equal.

***Remark* 2.1.** — We explain the comment about Serre duality in more detail. For an arbitrary holomorphic vector bundle E, we have a duality $H^q(X, E) = H^{n-q}(X, K \otimes E^*)^*$ and in particular, these vector spaces have the same dimension. When $E = \mathcal{H}^p$, this says

$$\dim H^q(X, \mathcal{H}^p) = \dim H^{n-q}(X, \mathcal{H}^{p*} \otimes K) \tag{1}$$

Now we have a perfect pairing $\mathcal{H}^p \otimes \mathcal{H}^{n-p} \to K$ given by wedge product. Tensoring the (tautological) perfect pairing $\mathcal{H}^p \otimes \mathcal{H}^{p*} \to \underline{\mathbb{C}}$ with K, we see that $\mathcal{H}^{p*} \otimes K$ is canonically isomorphic to $\mathcal{H}^{n-p}$, so the result follows from (1). **Editor's Note: Hunter has rendered this much clearer than it was in class.**

Proposition 2.2. *For any compact Kähler manifold, the odd-degree cohomology groups $H^{2k+1}(X, \mathbb{C})$ are even-dimensional.*

Proof. We have

$$H^{2k+1}(X, \mathbb{C}) = \bigoplus_{p+q=2k+1} H^q(X, \mathcal{H}^p).$$

The summands on the right hand side consist of pairs of equal-dimensional subspaces where p and q are switched, since $p \neq 2k+1-p$ for any p. □

The proposition shows, for instance, that there can be no Kähler structure on $S^3 \times S^1$, even though this manifold does admit a complex structure. Namely, take $q \in \mathbb{C}^\times$, with $|q| < 1$, and consider the quotient of $\mathbb{C}^2 \setminus \{(0,0)\}$ by the relation

$$\ldots \sim (z_1, z_2) \sim (qz_1, qz_2) \sim (q^2 z_1, q^2 z_2) \sim \ldots$$

A fundamental domain is $q^2 \leq |z_1|^2 + |z_2|^2 < 1$. This is $S^3 \times [0,1]$, so the quotient is homeomorphic to $S^3 \times S^1$. So this is a compact complex manifold which doesn't admit a Kähler structure at all.

Proposition 2.3. *For any compact Kähler manifold, $H^2(X, \mathbb{C})$ is nontrivial.*

Proof. We checked on the homework that for any Kähler manifold, ω is killed by d^*. So it's harmonic, and it's certainly nonzero (from the non-degeneracy of the associated Hermitian inner product). Writing $H^{p,q}(X)$ for the Harmonic (p,q)-forms, we see that $H^{1,1}(X)$ is nonzero. □

Thus we see that the 6-sphere S^6 doesn't admit a Kähler structure. Now we will show the result of Proposition 2.3 holds for *all* even-dimensional cohomology groups.

Lemma 2.4. *If ω is a Kähler form, then for $1 \leq k \leq n$, the k-form $\overbrace{\omega \wedge \omega \wedge \ldots \wedge \omega}^{k}$ is harmonic and nonzero.*

Proof. It follows from problem 2 on Problem Set 9 that L and Λ take harmonic forms to harmonic forms, and the form we're looking at is just L^k acting on the harmonic 0-form 1. It remains to show that the form is nonzero. It's enough to check this for $\underbrace{\omega \wedge \ldots \wedge \omega}_{n}$, since

$$\underbrace{\omega \wedge \ldots \wedge \omega}_{n} = \underbrace{\omega \wedge \ldots \wedge \omega}_{k} \wedge \underbrace{\omega \wedge \ldots \wedge \omega}_{n-k}.$$

We claim this is just $n!$ times the volume form, which is itself obviously nonzero. We can check this equality at a point. Let

$$z_j = x_j + iy_j$$

be nice coordinates centered at a given point. Then

$$\begin{aligned} \omega &= \frac{1}{-2i} \sum dz_j \wedge \overline{dz}_j + \ldots \\ &= \sum dx_j \wedge dy_j + \ldots, \text{ so} \\ \wedge^n \omega &= n!(dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n) + \ldots \end{aligned}$$

(We don't get any negative signs in the last line of the computation, because we always commute an even number of terms past an even number of terms.) Now, the higher order terms omitted in each line of the above computation vanish at the point $z_1, \ldots, z_n$, and the point at which we centered our coordinates was arbitrary, so $\frac{\wedge^n \omega}{n!}$ is equal on the nose to the volume form. □

The same proof as in Proposition 2.3 gives:

Corollary 2.5. *For X a compact Kähler manifold, we have $\dim H^{k,k}(X) \geq 1$. In particular, the even-degree cohomology groups $H^{2k}(X, \mathbb{C})$ are non-trivial.*

Later, we'll show something stronger than the proposition, namely, that

$$L^{n-k} : \text{ Harmonic } k\text{-forms} \to \text{ Harmonic } (2n-k)\text{-forms}$$

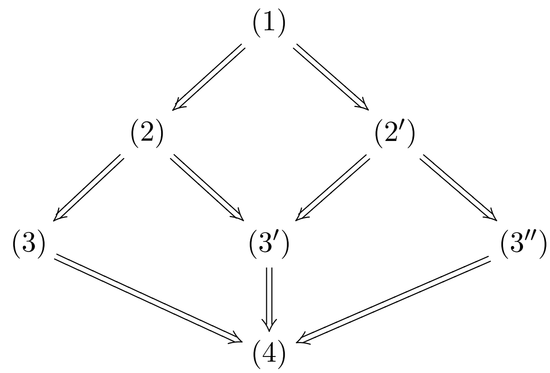
is an isomorphism.

3. The $\partial\overline{\partial}$-Lemma

The following is often called the $\partial\overline{\partial}$-lemma, and its name is often pronounced "dee dee bar" rather than "del del bar."

Lemma 3.1 (Reference: Voisin 6.17)**.** *Let X be compact Kähler. For α a d-closed (p,q)-form, the following are equivalent:*
(1)$\alpha = \partial\overline{\partial}\beta$ *for some* $(p-1, q-1)$*-form* β.
(2)$\alpha = \partial\gamma$ *for some* $\overline{\partial}$*-closed* γ.
(2′)$\alpha = \overline{\partial}\gamma'$ *for some* ∂*-closed* γ.
(3)$\alpha = \partial\gamma$ *for some* γ
(3′)$\alpha = d\gamma'$ *for some* γ'.
(3″)$\alpha = \overline{\partial}\gamma'$ *for some* γ'.
(4)$\alpha = \partial\gamma + \overline{\partial}\gamma'$ *for some* γ, γ'.

Proof. The easy implications, which have nothing to do with the compact or Kähler hypotheses, are depicted in the diagram below. For instance, to see that (1) implies (2), just take $\gamma = \overline{\partial}\beta$. The other implications all follow from (4) implies (1), which we'll do in the next lecture.



□

4. Independence of Hodge decomposition from Kähler structure

The remainder of this was presently poorly in class, and will be redone at the start of Thursday.

Using this lemma, we can explain why, for a compact Kähler manifold, the Hodge decomposition $H^k(X, \mathbb{C}) = \bigoplus H^{p,q}(X)$ is independent of the choice of Kähler structure. First, we claim that we have a decomposition

$$H^k_{\mathrm{dR}}(X, \mathbb{C}) = \bigoplus_{p+q=k} \frac{\{d\text{-closed } (p,q)\text{-forms}\}}{\{d\text{-exact } (p,q)\text{-forms}\}}.$$

Indeed, any closed k-form $\overline{\alpha}$ has a unique harmonic representative α, which decomposes into $\sum \alpha^{p,q}$, and, again using the Kähler relations, the $\alpha^{p,q}$ are harmonic. This decomposition is independent of the particular Kähler structure used to prove it, since it's just a statement that the de Rham cohomology classes are spanned by the (p,q)-subspaces (which have pairwise trivial intersection).

So what we need to show is that

$$\frac{\{d\text{-closed } (p,q)\text{-forms}\}}{\{d\text{-exact } (p,q)\text{-forms}\}} \cong H^q(X, \mathcal{H}^p).$$

independent of the choice of Kähler structure. We can do this by showing the two identifications

$$\frac{\{d\text{-closed } (p,q)\text{-forms}\}}{\{d\text{-exact } (p,q)\text{-forms}\}} \overset{(a)}{=} \frac{\{d\text{-closed } (p,q)\text{-forms}\}}{\overline{\partial}\{\partial\text{-closed}(p,q-1)\text{-forms.}\}} \overset{(b)}{\cong} \frac{\{\overline{\partial}\text{-closed } (p,q)\text{-forms}\}}{\overline{\partial}\{\text{all } (p,q-1)\text{forms}\}},$$

since this last space is just $H^{p,q}(X)$ (as follows from the $\overline{\partial}$-Poincare lemma). Identification (a) (which is literally an equality) follows because the two spaces being killed are the same, by the equivalence $(2') \Leftrightarrow (3')$ of the $\partial\overline{\partial}$-Lemma.

For equality (b), we certainly have a map

$$\frac{\{d\text{-closed } (p,q)\text{-forms}\}}{\overline{\partial}\{\partial\text{-closed}(p,q-1)\text{-forms.}\}} \to \frac{\{\overline{\partial}\text{-closed } (p,q)\text{-forms}\}}{\overline{\partial}\{\text{all } (p,q-1)\text{forms}\}}.$$

To see that this map is injective, suppose that a d-closed form α is of the form $\overline{\partial}\gamma'$ for some γ' a $(p,q-1)$-form. Then by $(3'') \Rightarrow (2')$ of the $\partial\overline{\partial}$-Lemma, we see that $\alpha = \overline{\partial}\gamma'$ for some ∂-closed γ, and hence is zero in the domain. To see that it is surjective, take an arbitrary $\overline{\partial}$-closed (p,q) form η, and set $\alpha = \partial\eta$. Then $d\alpha = \partial\partial\eta + \overline{\partial}\partial\eta = 0 - \partial\overline{\partial}\eta = 0$, and using $(2) \Rightarrow (1)$ of the $\partial\overline{\partial}$-Lemma, we see that $\alpha = \partial\overline{\partial}\beta$ for some β. Thus $\eta - \overline{\partial}\beta \in \ker \partial$ and likewise $\ker \overline{\partial}$, so is in $\ker d$; the image of $\eta - \overline{\partial}\beta$ under our map is (the class of) η.