# NOTES FOR MARCH 24

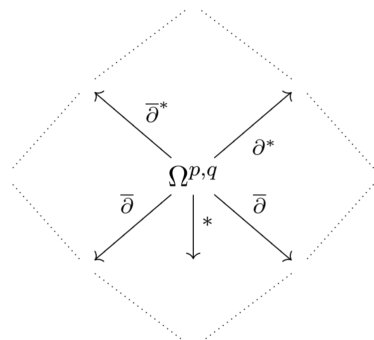
SCRIBE: GIWAN KIM

## 1. What does the Hodge decomposition look like?

Let $X$ be a (compact) complex manifold with a Hermitian metric. We have defined the operators

$$\partial, \overline{\partial}, *, \partial^*, \overline{\partial}^*$$

where the first two operators are defined in terms of complex structure, $*$ takes the metric structure as input and the two adjoint operators $\partial^*$ and $\overline{\partial}^*$ takes a combination of complex structure and the metric as input. On the Hodge diamond the operators can be visualized as




Some identities derived from $d = \partial + \overline{\partial}$ and $d^2 = 0$ are

- $\partial^2 = 0$
- $\overline{\partial}^2 = 0$
- $\partial\overline{\partial} + \overline{\partial}\partial = 0$

and corresponding adjoint operators satisfy

- $(\partial^*)^2 = 0$
- $(\overline{\partial}^*)^2 = 0$
- $\partial^*\overline{\partial}^* + \overline{\partial}^*\partial^* = 0$

deduced from $d^* = \partial^* + \overline{\partial}^*$ and $(\overline{\partial}^*)^2 = 0$.

We can compute the Laplacian $\Delta_d$ in terms of the "partial" operators:

$$\begin{aligned}\Delta_d &= (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) \\ &= (\partial\partial^* + \partial^*\partial) + (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}) + (\partial\overline{\partial}^* + \overline{\partial}^*\partial) + (\partial^*\overline{\partial} + \overline{\partial}\partial^*) \\ &= \Delta_\partial + \Delta_{\overline{\partial}} + \text{pairs of maps going sideways in the Hodge diamond.}\end{aligned}$$

Suppose $X$ is Kähler so that $\partial\overline{\partial}^* + \overline{\partial}^*\partial = 0$ and $\partial^*\overline{\partial} + \overline{\partial}\partial^* = 0$ "no sideways skipping" and $\Delta_d = \partial\partial^* + \partial^*\partial = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} = \Delta_{\overline{\partial}}$. Moreover, there is a suitable notion to make sense of the decomposition

$$\Omega^{p,q}(X) = \bigoplus_{\lambda \geqslant 0} (\lambda\text{-eigenspaces for } \Delta) = \bigoplus_{\lambda \geqslant 0} \Omega^{p,q}_\lambda(X)$$

where $\Delta := \Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d$ such that

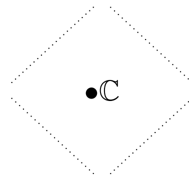
1. there is a (discrete) sequence $\{\lambda_i\}$ with $\lambda_i \geqslant 0$ and $\lambda \to \infty$
2. the $\lambda$-eigenspaces for $\Delta$, $\Omega^{p,q}_\lambda$ are finite dimensional
3. every $(p, q)$-form can be written uniquely as a sum of one element from each $\Omega^{p,q}_\lambda$.

Moreover, $\partial, \overline{\partial}, \partial^*, \overline{\partial}^*$ all commute with $\Delta$. For example,

$$\partial\Delta = \partial(\partial\partial^* + \partial^*\partial) = \partial\partial^*\partial = (\partial\partial^* + \partial^*\partial)\partial = \Delta\partial.$$

So $\partial, \overline{\partial}, \partial^*, \overline{\partial}^*$ preserves the eigenspace decomposition; i.e. maps $\Omega_\lambda^{\bullet,\bullet}$ to $\Omega_\lambda^{\bullet,\bullet}$.

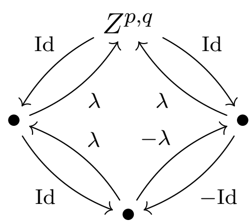
On $\Omega_0^{\bullet,\bullet}$, the harmonic forms $\partial, \overline{\partial}, \partial^*, \overline{\partial}^*$ are all zero maps. So on the Hodge diamond harmonic forms are direct sums of




As operators on $\Omega_\lambda^{\bullet,\bullet}$ for $\lambda > 0$, $\partial, \overline{\partial}, \partial^*, \overline{\partial}^*$ satisfy

(1) $\partial, \overline{\partial}^2, (\partial^*)^2, (\overline{\partial}^*)^2$ are 0
(2) $\partial\overline{\partial} + \overline{\partial}\partial, \partial\overline{\partial}^* + \overline{\partial}^*\partial, \partial^*\overline{\partial}^* + \overline{\partial}^*\partial^*, \overline{\partial}\partial^* + \partial^*\overline{\partial}$ are 0
(3) $\partial\partial^* + \partial^*\partial = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} = \lambda \mathrm{Id}$.

Let $Z^{p,q}$ be the subspace of $\Omega_\lambda^{p,q}$ killed by $\partial^*$ and $\overline{\partial}^*$. On the homework, we showed that $Z^{p,q} \cong \partial Z^{p,q} \cong \overline{\partial} Z^{p,q} \cong \partial\overline{\partial} Z^{p,q}$ and that we can choose bases such that the maps between the four spaces are

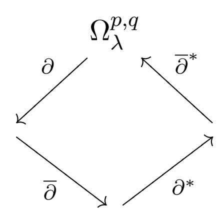



We also showed that

$$\Omega_\lambda^{p,q} = Z^{p,q} \oplus \partial Z^{p-1,q} \oplus \overline{\partial} Z^{p,q-1} \oplus \partial\overline{\partial} Z^{p-1,q-1}.$$

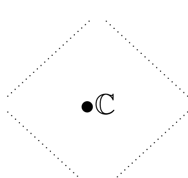
One way to do this is to write down formulas for the four projections $\Pi_i : \Omega_\lambda^{p,q} \longrightarrow \Omega_\lambda^{p,q}$ such that $\Pi_i \Pi_j = \delta_{ij}$ and $\Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 = \mathrm{Id}$. For instance, one can define the projection onto $Z^{p,q}$ as

$$\Pi_1 = -\frac{1}{\lambda^2}\overline{\partial}^*\partial^*\overline{\partial}\partial = -\frac{1}{\lambda^2}$$

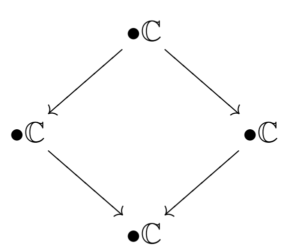



Similarly, the other maps involve going in circles in the Hodge diamond and that the $\Pi_i$ satisfy the relations can be seen (without writing anything down) by thinking about the relations among $\partial, \overline{\partial}, \partial^*, \overline{\partial}^*$ in terms of pictures. For instance, going in circle counter-clockwise is the same as going clockwise since sign reverses twice....

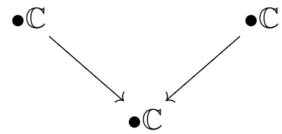
As a consequence, $\Omega^{\bullet,\bullet}$ decomposes into a direct sum of




and



On a non-Kähler manifold such as the Hopf surface ($\cong S^1 \times S^3$) the "angle" form on $S^1$ pulls back to a combination of $(1,0)$ and $(0,1)$-form which fits into the diagram



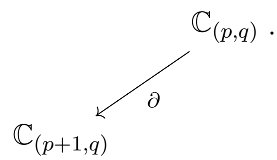

which should not appear on Kähler manifolds.

We have the Hodge decomposition

$$\frac{\operatorname{Ker}(\partial + \overline{\partial} : \Omega^k \longrightarrow \Omega^{k+1})}{\operatorname{Im}(\partial + \overline{\partial} : \Omega^{k-1} \longrightarrow \Omega^k)} \cong \bigoplus_{p+q=k} \frac{\operatorname{Ker}(\overline{\partial} : \Omega^{p,q} \longrightarrow \Omega^{p,q+1})}{\operatorname{Im}(\overline{\partial} : \Omega^{p,q-1} \longrightarrow \Omega^{p,q})}.$$

To see Hodge Theorem (concerning harmonic representatives for deRham or Dolbeault cohomology) from what we have discussed so far notice that $(p,q)$-harmonic forms contribute to LHS when $k = p+q$ and contributes to $(p,q)$-summand on RHS, while a $2 \times 2$ square contributes nothing to both sides.

It is also interesting to see how you could have $\dim H^k(X)$ different from $\sum_{p+q=k} \dim H^q(X, \mathcal{H}^p)$. For example, we could have a picture like:




(Here all other maps are zero and $\mathbb{C}_{(p,q)}$ denotes an 1-dimensional subspace of $\Omega^{p,q}$.) This contributes nothing to the LHS but on the RHS contributes to the $(p,q)$ and $(p+1,q)$-summand.

## 2. Finishing up the $\partial\overline{\partial}$ lemma

Missing part of $\partial\overline{\partial}$-lemma was (4) implies (1); that is, if $\alpha$ is a $(p,q)$-form which is $\partial\gamma_1 + \overline{\partial}\gamma_2$ and $\alpha$ is $d$-closed then $\alpha = \partial\overline{\partial}\beta$. Since $\alpha$ is $d$-closed, $\alpha$ is a sum of harmonic forms and bottom of squares. Image of $\partial$ is lower left side of these squares and image of $\overline{\partial}$ is lower right side of the squares. So $\alpha$ must be a sum of bottom parts of squares which implies that $\alpha$ is in the image of $\partial\overline{\partial}$.

With the $\partial\overline{\partial}$-lemma we can prove the independence of Hodge decomposition from Kähler structure. This has already been written down in the notes for March 22.

**Remark:** Every double complex over a field is a direct sum of squares, dots, zigzags. So the interesting fact is that there are no zig-zags.

## 3. Standard Coordinates on $\mathbb{C}^n$

There is a lot of sign confusion, so here is a formula you can memorize.

Let $z_1, \ldots, z_n$ be the usual complex coordinates on $\mathbb{C}^n$ so that $z_j = x_j + iy_j$. We can write the standard Kähler form in coordinates as

$$\omega = \frac{1}{(-2i)^n} \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

Let $A_j = dx_j \wedge dy_j = \frac{1}{-2i} dz_j \wedge d\overline{z}_j$. On $\mathbb{C}$, we have $*1 = A_1$, $*A_1 = 1$, $*(dz) = *(dx + idy) = dy - idx = -idz$, and $*(d\overline{z}) = id\overline{z}$. Write $\eta = \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_n$ where $\eta_j \in \{1, dz_j, d\overline{z}_j, A_j\}$ and let $a$ be the number of occurrences of 1, $b$ for $dz_j$, $c$ for $d\overline{z}_j$, and $d$ for $A_j$. Then

$$*\eta = (-1)^{\binom{b+c}{2}} (*_1\eta_1) \wedge (*_2\eta_2) \wedge \cdots \wedge (*_n\eta_n).$$